lowing nonzero strain components:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \frac{\nu_{12}}{E_1}\sigma_{22}$$
$$\varepsilon_{22} = \frac{\sigma_{22}}{E_2} - \frac{\nu_{12}}{E_1}\sigma_{11}$$
$$\varepsilon_{33} = -\frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22}$$

Using the maximum strain failure criterion and substituting $\sigma_{11} = \sigma_0$, we have

$$-\frac{X}{E_1} < \frac{\sigma_0}{E_1} - \frac{\nu_{12}}{E_1}\sigma_{22} < \frac{X}{E_1}$$
$$-\frac{Y}{E_2} < \frac{\sigma_{22}}{E_2} - \frac{\nu_{12}}{E_1}\sigma_0 < \frac{Y}{E_2}$$
$$-\frac{Z}{E_3} < -\frac{\nu_{13}}{E_1}\sigma_0 - \frac{\nu_{23}}{E_2}\sigma_{22} < \frac{Z}{E_3}$$

which can be reduced to be

$$\frac{\sigma_0 - X}{\nu_{12}} < \sigma_{22} < \frac{X + \sigma_0}{\nu_{12}}$$
$$-Y + \frac{\nu_{12}E_2}{E_1}X < \sigma_{22} < Y + \frac{\nu_{12}E_2}{E_1}X$$
$$-\frac{\nu_{13}E_2}{\nu_{23}E_1}X - \frac{ZE_2}{\nu_{23}E_3} < \sigma_{22} < -\frac{\nu_{13}E_2}{\nu_{23}E_1}X + \frac{ZE_2}{\nu_{23}E_3}$$

Because Poisson's ratios are all positive, we can conclude

$$\max\left(\frac{\sigma_0 - X}{\nu_{12}}, -Y + \frac{\nu_{12}E_2}{E_1}X, -\frac{\nu_{13}E_2}{\nu_{23}E_1}X - \frac{ZE_2}{\nu_{23}E_3}\right) < \sigma_{22} < \min\left(\frac{X + \sigma_0}{\nu_{12}}, Y + \frac{\nu_{12}E_2}{E_1}X, -\frac{\nu_{13}E_2}{\nu_{23}E_1}X + \frac{ZE_2}{\nu_{23}E_3}\right)$$

The possible values of σ_{22} for which the material does not fail depend on σ_0 and Young's moduli, and Poisson's ratios in addition to the strength parameters. This result is clearly different from what one could obtain from the maximum stress failure criterion which is

$$-Y < \sigma_{22} < Y$$

In general, the maximum stress criterion and the maximum strain criterion will predict different results even if the material is linear elastic up to the failure.

6.3.2 Tsai-Hill failure criterion

Maximum stress (strain) criterion applies the failure criterion to individual stress components. The clear inconsistency is that the corresponding strengths are measured under uniaxial stress states by designing the experiments such that only one stress component exists in the material. However the material in real structures is usually subjected to a multi-axial stress state with the possibility that all six stress components exist. By subjecting individual stress components in Eqs. (6.19), (6.20), and (6.21), this failure criterion completely neglects the interaction among different stress components. For example, a material fails under uniaxial stress state when $\sigma_{11} = X$ or $\sigma_{22} = Y$, however when it is subjected to the biaxial loading $\sigma_{11} = X$ and $\sigma_{22} = Y$ simultaneously, the material may have already failed or is still safe depending on the material.

As a remedy for this deficiency, Hill [61] extended the Mises failure criterion to orthotropic materials such that

$$f = F(\sigma_{22}-\sigma_{33})^2+G(\sigma_{33}-\sigma_{11})^2+H(\sigma_{11}-\sigma_{22})^2+L\sigma_{23}^2+M\sigma_{13}^2+N\sigma_{12}^2 = 1 \quad (6.22)$$

with F, G, H, L, M, N as strength parameters for this failure criterion which are calibrated by experiments. Tsai [62] applied this failure criterion to UDFRCs by calibrating these parameters from tests of the normal strengths in three material principal directions and shear strengths in three orthogonal planes of symmetry. Because of its application of composites, we thus call this failure criterion as the Tsai-Hill failure criterion. This criterion assumes that the material has the same normal strength in both tension and compression. By applying this failure criterion to simple tension tests and shear tests when only one stress component is equal to its corresponding strength and all the other stress components vanish, we have

$$\begin{aligned}
(G+H)X^2 &= 1 \qquad \text{for } \sigma_{11} = X \text{ and other components vanish} \\
(F+H)Y^2 &= 1 \qquad \text{for } \sigma_{22} = Y \text{ and other components vanish} \\
(F+G)Z^2 &= 1 \qquad \text{for } \sigma_{33} = Z \text{ and other components vanish} \\
LR^2 &= 1 \qquad \text{for } \sigma_{23} = R \text{ and other components vanish} \\
MT^2 &= 1 \qquad \text{for } \sigma_{13} = T \text{ and other components vanish} \\
NS^2 &= 1 \qquad \text{for } \sigma_{12} = S \text{ and other components vanish} \qquad (6.23)
\end{aligned}$$

which can be used to determine the following

$$\begin{aligned}
2F &= \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \\
2G &= \frac{1}{X^2} + \frac{1}{Z^2} - \frac{1}{Y^2} \\
2H &= \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \\
L &= \frac{1}{R^2} \\
M &= \frac{1}{T^2} \\
N &= \frac{1}{S^2} \qquad (6.24)
\end{aligned}$$

Having measured the tensile strengths in three material principal directions and shear strengths in the three planes of orthotropic symmetry, we can evaluate the failure function f in Eq. (6.22) with the strength parameters determined by Eq. (6.24) to check whether the material fails under a general stress state σ_{ij}.

6.3.2.1 Plane-stress reduced Tsai-Hill failure criterion For a thin composite laminate, we commonly assume that the stress state is plane-stress which implies $\sigma_{i3} = 0$. The Tsai-Hill failure criterion in Eq. (6.22) is simplified to be

$$f = (G+H)\sigma_{11}^2 + (F+H)\sigma_{22}^2 - 2H\sigma_{11}\sigma_{22} + N\sigma_{12}^2 = 1 \qquad (6.25)$$

with $G+H = \frac{1}{X^2}, F+H = \frac{1}{Y^2}, N = \frac{1}{S^2}$. If we further assume $Z = Y$, which implies that the tensile strengths along the two transverse directions of a unidirectional fiber reinforced composite are equal, we have $2H = \frac{1}{X^2}$ according to Eq. (6.24). The Tsai-Hill failure criterion can be rewritten as

$$f = \left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 - \frac{\sigma_{11}}{X}\frac{\sigma_{22}}{X} + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.26}$$

Recall that we have assumed that the tensile strength is equal to compressive strength along three principal material directions. This does not agree with what we measure from most fiber reinforced composites. In other words, we will have a total of four normal strengths (two tensile and two compressive, X, Y and X', Y') for the plane-stress state. In some applications of the Tsai-Hill failure criterion in Eq. (6.26), we substitute tensile strength to positive normal stress and compressive strength to negative normal stress. For example, if the stress state is $\sigma_{11} > 0$ and $\sigma_{22} < 0$, the Tsai-Hill failure criterion is expressed as

$$f = \left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y'}\right)^2 - \frac{\sigma_{11}}{X}\frac{|\sigma_{22}|}{X'} + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.27}$$

6.3.2.2 Equivalence of Tsai-Hill criterion and Mises criterion for isotropic materials

We also want to examine whether the Tsai-Hill failure criterion is equivalent to the Mises failure criterion for isotropic materials. For isotropic materials, we have $X = Y = Z, R = S = T$. Thus, the 3D Tsai-Hill failure function for isotropic materials can be expressed as

$$\begin{aligned} f &= \frac{(\sigma_{22}-\sigma_{33})^2}{2X^2} + \frac{(\sigma_{33}-\sigma_{11})^2}{2X^2} + \frac{(\sigma_{11}-\sigma_{22})^2}{2X^2} + \frac{\sigma_{23}^2}{S^2} + \frac{\sigma_{13}^2}{S^2} + \frac{\sigma_{12}^2}{S^2} \\ &= \frac{1}{2X^2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + \frac{2X^2}{S^2}\left(\sigma_{23}^2+\sigma_{13}^2+\sigma_{12}^2\right)\right] \end{aligned} \tag{6.28}$$

For it to be equivalent to the Mises failure criterion in Eq. (6.10), we require $\frac{2X^2}{S^2} = 6$ or $X = \sqrt{3}S$, which is the condition for the Mises criterion to be equivalently calibrated either using the simple tension test or the simple shear test.

EXAMPLE 6.6

Strength analysis of a composite lamina: Suppose that a composite layer can be assumed as a homogeneous material. The composite material has tensile strength along fiber direction as X=100 ksi, tensile strength along transverse direction as Y=9 ksi, and in-plane shear strength as S=15 ksi. The material behaves linearly elastic up to failure. An off-axis lamina is loaded by a tensile σ_0 as shown in Figure 6.10. Predict the maximum allowable σ_0 as a function of the fiber orientation angle θ_3 according to the maximum stress criterion, the maximum strain criterion, and the Tsai-Hill criterion.

Solution: First, we need to evaluate the stress components in the material coordinate system (x_i) since σ_0 is in the problem coordinate system x_i'. We have

$$\begin{aligned} \sigma_{11} &= \cos^2\theta_3\sigma_0 \\ \sigma_{22} &= \sin^2\theta_3\sigma_0 \\ \sigma_{12} &= -\cos\theta_3\sin\theta_3\sigma_0 \end{aligned}$$

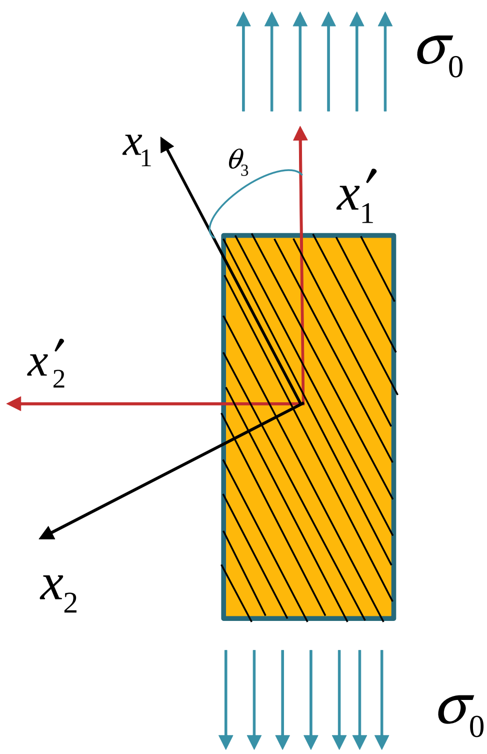

Figure 6.10 An off-axis lamina loaded by uniaxial tension.

According to the maximum stress criterion, the lamina fails when $\sigma_{11} = X, \sigma_{22} = Y, |\sigma_{12}| = S$. We have

$$\begin{aligned} \sigma_0 &= \frac{X}{\cos^2\theta_3} \\ \sigma_0 &= \frac{Y}{\sin^2\theta_3} \\ \sigma_0 &= \frac{S}{\cos\theta_3 \sin\theta_3} \end{aligned} \tag{6.29}$$

σ_0 must be the smallest among the above three values to satisfy the maximum stress failure criterion. If the first value is the smallest, the lamina fails in the fiber direction (fiber failure), if the second value is the smallest, the lamina fails in the transverse direction (matrix failure), if the third value is the smallest, the lamina fails due to in-plane shear (matrix failure). It is noted that because θ_3 is defined between $0°$ and $90°$, $\cos\theta_3 \sin\theta_3 > 0$.

To use the maximum strain criterion, we need to compute strains in the material coordinate system first as

$$\begin{aligned} \varepsilon_{11} &= \frac{\sigma_0}{E_1}\left(\cos^2\theta_3 - \nu_{12}\sin^2\theta_3\right) \\ \varepsilon_{22} &= \frac{\sigma_0}{E_2}\left(\sin^2\theta_3 - \nu_{21}\cos^2\theta_3\right) \\ 2\varepsilon_{12} &= -\frac{\sigma_0}{G_{12}}\cos\theta_3 \sin\theta_3 \end{aligned}$$

According to the maximum strain criterion, the lamina fails when $\varepsilon_{11} = X_\varepsilon, \varepsilon_{22} = Y_\varepsilon, |2\varepsilon_{12}| = S_\varepsilon$. Since the material is linear elastic up to failure, we have the allowable strains as $X_\varepsilon = \frac{X}{E_1}, Y_\varepsilon = \frac{Y}{E_2}, S_\varepsilon = \frac{S}{G_{12}}$. Using this fact, we have the following three equations according to the maximum strain criterion.

$$\begin{aligned} \sigma_0 &= \frac{X}{\cos^2\theta_3 - \nu_{12}\sin^2\theta_3} \\ \sigma_0 &= \frac{Y}{\sin^2\theta_3 - \nu_{21}\cos^2\theta_3} \\ \sigma_0 &= \frac{S}{\cos\theta_3 \sin\theta_3} \end{aligned} \tag{6.30}$$

Again, σ_0 must be the smallest among the above three values to satisfy the maximum strain failure criterion. If the first value is the smallest, the lamina fails in the fiber direction (fiber failure), if the second value is the smallest, the lamina fails in the transverse direction (matrix failure), if the third value is the smallest, the lamina fails due to in-plane shear (matrix failure).

Lastly, according to the Tsai-Hill criterion in Eq. (6.26), we have

$$\left(\frac{\cos^2\theta_3\sigma_0}{X}\right)^2 + \left(\frac{\sin^2\theta_3\sigma_0}{Y}\right)^2 - \left(\frac{\cos\theta_3\sin\theta_3\sigma_0}{X}\right)^2 + \left(\frac{\cos\theta_3\sin\theta_3\sigma_0}{S}\right)^2 = 1$$

which can be solved as

$$\sigma_0 = \frac{1}{\sqrt{\left(\frac{\cos^2\theta_3}{X}\right)^2 + \left(\frac{\sin^2\theta_3}{Y}\right)^2 - \left(\frac{\cos\theta_3\sin\theta_3}{X}\right)^2 + \left(\frac{\cos\theta_3\sin\theta_3}{S}\right)^2}} \tag{6.31}$$

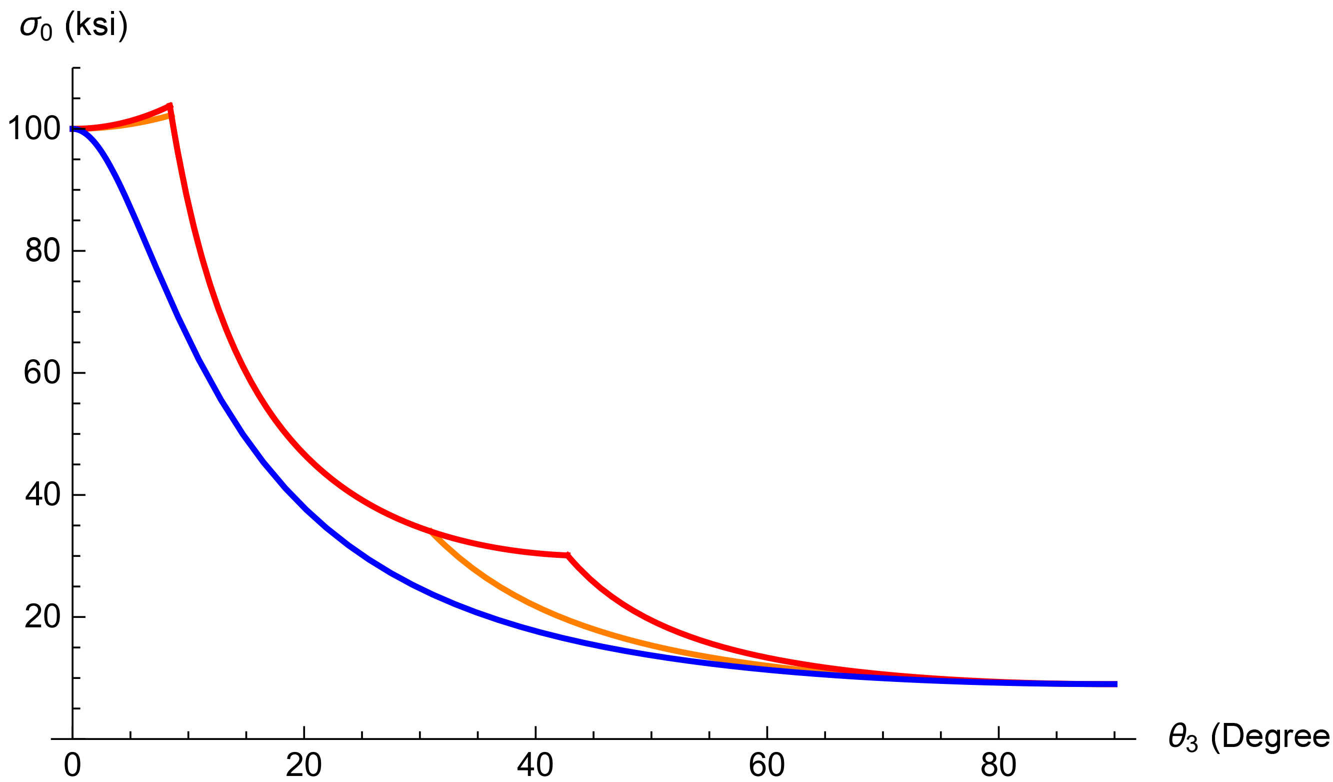

Figure 6.11 Strength of an off-axis lamina predicted by different failure criteria (Orange - maximum stress criterion; Red - maximum strain criterion; Blue - Tsai-Hill criterion).

The predicted strength σ_0 as a function of the fiber orientation angle θ_3 in Eqs. (6.29), (6.30) (to plot these equations, we have assumed $\nu_{12} = 0.7$ and $\nu_{21} = 0.3$), and (6.31) can be plotted in Figure 6.11. As shown in the plot, the results predicted by the Tsai-Hill failure criterion is the most conservative for this case, followed by the maximum stress failure criterion, and then the maximum strain failure criterion. The allowable σ_0 predicted by the Tsai-Hill failure criterion is a continuous curve while the two other criteria predicted piecewise continuous curves. There are very small differences between the maximum stress criterion and the maximum strain criterion when the fiber orientation is smaller than $31°$. As indicated by the maximum stress failure criterion and the maximum strain failure criterion, with the angle increasing from $0°$ to $90°$, the lamina fails in the fiber direction first, then fails by in-plane shear, and finally fails in the transverse direction.

EXAMPLE 6.7

A $[\pm 45/0/90]_s$ laminate made of composite layers with lamina constants $E_1 = 20 \times 10^6$ psi, $E_2 = 1.5 \times 10^6$ psi, $G_{12} = 10^6$ psi, $\nu_{12} = 0.29, X = 310$ ksi, Y=9 ksi, and S=15 ksi. The thickness of each layer is 0.005”. This laminate is subject to N_{11}. Assume that the composite material fails according to the Tsai-Hill failure criterion.

Compute the maximum allowable N_{11} before any of the layers fail. Which layer fails first?

Solution: First, we need to evaluate the plate stiffness of the laminate. Because it is a symmetric laminate loaded only by the in-plane load N_{11}, the A matrix is sufficient for the analysis. As we have learned previously, we can obtain the plane-stress reduced stiffness matrix Q for the composite lamina, then transform Q according to the layup orientation, and integrate the transformed Q matrices through the thickness to obtain A matrix as

$$A = \begin{bmatrix} 348925 & 101315 & 0 \\ 101315 & 348925 & 0 \\ 0 & 0 & 123805 \end{bmatrix} \text{lb/in}$$

We can obtain the in-plane plate strains due to the applied load N_{11} as

$$\epsilon = A^{-1} \begin{Bmatrix} N_{11} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3.12983 \times 10^{-6} \\ -9.0879 \times 10^{-7} \\ 0 \end{Bmatrix} N_{11}\text{in/lb}$$

Since $\kappa = 0$, the 3D in-plane strains are $\varepsilon_e = \epsilon$. Next, we need to compute the 3D stresses for each layer. We need to first use $\sigma'_e = Q'\varepsilon_e$ (Q' is the Q matrix for each layer in the laminate coordinate system) to compute 3D stresses in the laminate coordinate system, then we need to transform the 3D stresses into the material coordinate system using $\sigma_e = R_{\sigma e}^{-1}\sigma'_e$. Following this procedure, we obtain

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{\pm 45} = \begin{Bmatrix} 22.8375 \\ 2.16249 \\ \mp 4.03862 \end{Bmatrix} N_{11}/\text{in}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{0} = \begin{Bmatrix} 62.5961 \\ -0.00172 \\ 0 \end{Bmatrix} N_{11}/\text{in}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{90} = \begin{Bmatrix} -16.921 \\ 4.32671 \\ 0 \end{Bmatrix} N_{11}/\text{in}$$

Next, we need to plug the 3D stresses into the Tsai-Hill failure criterion for each layer, to compute the maximum allowable N_{11}. For the 45° and -45° layers, according to Eq. (6.26), we have

$$\left(\frac{22.8375N_{11}}{X}\right)^2 + \left(\frac{2.16249N_{11}}{Y}\right)^2 - \frac{22.8375 \times 2.16249N_{11}^2}{X^2} + \left(\frac{4.03862N_{11}}{S}\right)^2 = 1$$

which can be solved as $N_{11} = \pm 2720$ lb/in.

For the 0° layers, we have

$$\left(\frac{62.5961N_{11}}{X}\right)^2 + \left(\frac{-0.00172N_{11}}{Y}\right)^2 - \frac{22.8375 \times (-0.00172)N_{11}^2}{X^2} = 1$$

which can be solved as $N_{11} = \pm 4952$ lb/in.

For the 90° layers, we have

$$\left(\frac{16.921N_{11}}{X}\right)^2 + \left(\frac{-4.32671N_{11}}{Y}\right)^2 - \frac{16.921 \times (-4.32671)N_{11}^2}{X^2} = 1$$

which can be solved as $N_{11} = \pm 2063$ lb/in. When we increase the load (tensile or compressive) from 0, the 90° layers will fail first at $N_{11} = \pm 2063$ lb/in.

6.3.3 Tsai-Wu failure criterion

Generally speaking, if a failure criterion is stress based, we can express the failure function as

$$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) = 1 \tag{6.32}$$

A simple yet general enough failure criterion which can account for the difference between tensile and compressive strengths and interactions among different stress components for anisotropic materials was proposed by Tsai and Wu [63]. This criterion, commonly called the Tsai-Wu failure criterion for obvious reasons, can be written in a matrix form as below

$$f = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}^T \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}^T \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{12} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{13} & F_{23} & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{14} & F_{24} & F_{34} & F_{44} & F_{45} & F_{46} \\ F_{15} & F_{25} & F_{35} & F_{45} & F_{55} & F_{56} \\ F_{16} & F_{26} & F_{36} & F_{46} & F_{56} & F_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = 1 \tag{6.33}$$

It is noted that one can always form a symmetric matrix for the quadratic part of the failure criterion without changing the failure criterion. There are a total of 27 material parameters (F_i, F_{ij}) to be determined for a general anisotropic material.

Because we assume that the strength of the material does not depend on the sign of shear stresses, the strength parameters should remain the same when the material is subject to a positive shear stress or a negative shear stress. If we assume that the material is subjected to σ_{12} and all other stress components vanish, we have

$$f = F_6\sigma_{12} + F_{66}\sigma_{12}^2 \tag{6.34}$$

Since the strength parameters should remain the same when the material is subject to $-\sigma_{12}$ and all other stress components vanish, we will have $F_6 = 0$. Similarly, we can conclude that $F_4 = F_5 = 0$. Now, let us assume that the material is subjected to a stress state σ_{ij} with all six stress components existing, we obtain the failure function as f. If we change the stress state to be σ_{ij}^* so that all the stress components remain the same except we flip the sign of shear stress σ_{12} to be $\sigma_{12}^* = -\sigma_{12}$, we obtain the failure function as f^*. Since we have assumed that failure is independent of the sign of shear stress, we should have $f - f^* = 0$, which will help us obtain the following

$$4\sigma_{12}(F_{16}\sigma_{11} + F_{26}\sigma_{22} + F_{36}\sigma_{33} + F_{46}\sigma_{23} + F_{56}\sigma_{13}) = 0 \tag{6.35}$$

Since this equality should be valid for arbitrary values of the stress components involved in this equation, we can conclude that $F_{16} = F_{26} = F_{36} = F_{46} = F_{56} = 0$. Similarly, we can conclude $F_{15} = F_{25} = F_{35} = F_{45} = F_{14} = F_{24} = F_{34} = 0$.

With these conclusions, we can simplify the failure criterion in Eq. (6.33) to be

$$\begin{aligned} f =& F_1\sigma_{11} + F_2\sigma_{22} + F_3\sigma_{33} + F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + F_{33}\sigma_{33}^2 + 2F_{12}\sigma_{11}\sigma_{22} \\ &+ 2F_{13}\sigma_{11}\sigma_{33} + 2F_{23}\sigma_{22}\sigma_{33} + F_{44}\sigma_{23}^2 + F_{55}\sigma_{13}^2 + F_{66}\sigma_{12}^2 = 1 \end{aligned} \tag{6.36}$$

There are a total of 12 parameters in this failure criterion. Nine of these parameters can be calibrated by simple tension, compression, and shear tests of orthotropic materials. Assuming that we have measured the tensile strength along x_1 direction as X, the failure criterion in Eq. (6.36) should be able to describe the failure due to this simple stress state with $\sigma_{11} = X$, and all other stress components equal to zero. Substituting this stress state into Eq. (6.36), we have

$$F_1 X + F_{11} X^2 = 1 \tag{6.37}$$

Assuming that we have measured the compressive strength along x_1 direction as X', the failure criterion in Eq. (6.36) should be able to describe the failure due to this simple stress state with $\sigma_{11} = -X'$, and all other stress components equal to zero. Substituting this stress state into Eq. (6.36), we have

$$-F_1 X' + F_{11} X'^2 = 1 \tag{6.38}$$

From Eqs. (6.37) and (6.38), we can determine F_1, F_{11} as

$$F_1 = \frac{1}{X} - \frac{1}{X'}, \qquad F_{11} = \frac{1}{XX'} \tag{6.39}$$

Similarly for simple tension and compression tests along x_2 and x_3 directions, we obtain

$$F_2 = \frac{1}{Y} - \frac{1}{Y'}, \qquad F_{22} = \frac{1}{YY'}, \qquad F_3 = \frac{1}{Z} - \frac{1}{Z'}, \qquad F_{33} = \frac{1}{ZZ'} \tag{6.40}$$

For simple shear tests along the three planes of symmetry of the orthotropic material, we have

$$F_{44} = \frac{1}{R^2}, \qquad F_{55} = \frac{1}{T^2}, \qquad F_{66} = \frac{1}{S^2} \tag{6.41}$$

There are three more parameters F_{12}, F_{13}, F_{23} left to be determined. Fundamentally speaking, these constants should be determined from tests featuring bi-axial stress states. For example, to determine F_{12}, we should use a test featuring a stress state of σ_{11} and σ_{22}. Suppose we have tested the material and it fails under $\sigma_{11} = a$ and $\sigma_{22} = b$ and all the other stress components vanish. According to the Tsai-Wu failure criterion in Eq. (6.36), we have

$$F_1 a + F_2 b + F_{11} a^2 + 2F_{12} ab + F_{22} b^2 = 1 \tag{6.42}$$

We obtain

$$2F_{12} = \frac{1 - (F_1 a + F_2 b + F_{11} a^2 + F_{22} b^2)}{ab} \tag{6.43}$$

Since there are infinitely many such combined stress states which can break the material and it is impossible to prove the uniqueness of F_{12} from the above equation, we could end up with many values for F_{12}. To avoid arbitrariness, we introduce another method to determine F_{12}, F_{13}, F_{23} by requiring that the Tsai-Wu failure criterion reduces to be the Tsai-Hill failure criterion if the tensile strength is equal to the compressive strength in the same direction along three directions (i.e., $X = X', Y = Y', Z = Z'$). Clearly, under this condition, we have $F_1 = F_2 = F_3 = 0$ and $F_{11} = \frac{1}{X^2}, F_{22} = \frac{1}{Y^2}, F_{33} = \frac{1}{Z^2}$. The Tsai-Wu failure criterion becomes

$$\begin{aligned} f^* =& \frac{\sigma_{11}^2}{X^2} + \frac{\sigma_{22}^2}{Y^2} + \frac{\sigma_{33}^2}{Z^2} + \frac{\sigma_{23}^2}{R^2} + \frac{\sigma_{13}^2}{T^2} + \frac{\sigma_{12}^2}{S^2} + \\ & 2F_{12}\sigma_{11}\sigma_{22} + 2F_{13}\sigma_{11}\sigma_{33} + 2F_{23}\sigma_{22}\sigma_{33} \end{aligned} \tag{6.44}$$

Assume that the Tsai-Hill failure function is f, we have

$$\begin{aligned} 0 = f^* - f =& 2\sigma_{11}\left[F_{12} + H)\sigma_{22} + (F_{13} + G)\sigma_{33}\right] + \\ & 2\sigma_{22}\left[(F_{12} + H)\sigma_{11} + (F_{23} + F)\sigma_{33})\right] + \\ & 2\sigma_{33}\left[(F_{13} + G)\sigma_{11} + (F_{23} + F)\sigma_{22})\right] \end{aligned} \tag{6.45}$$

Since the above equality holds for all combination of $\sigma_{11}, \sigma_{22}, \sigma_{33}$, we have

$$
\begin{aligned}
2F_{12} = -2H &= \frac{1}{Z^2} - \frac{1}{X^2} - \frac{1}{Y^2} \\
2F_{13} = -2G &= \frac{1}{Y^2} - \frac{1}{X^2} - \frac{1}{Z^2} \\
2F_{23} = -2F &= \frac{1}{X^2} - \frac{1}{Y^2} - \frac{1}{Z^2}
\end{aligned} \tag{6.46}
$$

Since the main motivation of the Tsai-Wu failure criterion is to consider the differences between tensile and compressive strengths, we can modify Eq. (6.46) to be

$$
\begin{aligned}
2F_{12} &= \frac{1}{ZZ'} - \frac{1}{XX'} - \frac{1}{YY'} \\
2F_{13} &= \frac{1}{YY'} - \frac{1}{XX'} - \frac{1}{ZZ'} \\
2F_{23} &= \frac{1}{XX'} - \frac{1}{YY'} - \frac{1}{ZZ'}
\end{aligned} \tag{6.47}
$$

Note that the strength parameters F_{12}, F_{13}, F_{23} obtained using Eq. (6.47) still satisfy the requirement we used to derive Eq. (6.46) (i.e., the Tsai-Wu failure criterion can be reduced to the Tsai-Hill failure criterion when $X = X', Y = Y', Z = Z'$).

Until now we have determined all the 12 parameters in the Tsai-Wu failure criterion using the nine strengths obtained from 3 extension tests, 3 compression tests, and 3 shear tests. The Tsai-Wu failure criterion with parameters given in Eqs. (6.39), (6.40), (6.41) and (6.47) has the capability to account for different tensile and compressive strengths, and interaction among different stress components. It can be reduced to the Tsai-Hill failure criterion if the tensile and compressive strengths are assumed to be equal.

If we invoke the plane-stress assumption in the plane of $x_1 - x_2$ as we did in CLT, we will have $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. The Tsai-Wu failure criterion can be further simplified to be:

$$
f = F_1\sigma_{11} + F_2\sigma_{22} + F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + 2F_{12}\sigma_{11}\sigma_{22} + F_{66}\sigma_{12}^2 \tag{6.48}
$$

with F_1, F_{11} given in Eq. (6.39), F_2, F_{22} given in Eq. (6.40), F_{66} given in Eq. (6.41), F_{12} given in Eq. (6.47). If we further assume $Z = Y, Z' = Y'$, we have $2F_{12} = -\frac{1}{XX'}$. The Tsai-Wu failure criterion under these simplifications can be written explicitly in terms of strength parameters as

$$
f = \left(\frac{1}{X} - \frac{1}{X'}\right)\sigma_{11} + \left(\frac{1}{Y} - \frac{1}{Y'}\right)\sigma_{22} + \frac{\sigma_{11}^2}{XX'} + \frac{\sigma_{22}^2}{YY'} - \frac{\sigma_{11}}{X}\frac{\sigma_{22}}{X'} + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.49}
$$

When we assume that the compressive strengths are equal to the tensile strength, the plane-stress reduced Tsai-Wu failure criterion is the same as the plane-stress reduced Tsai-Hill failure criterion shown in Eq. (6.26).

EXAMPLE 6.8

A $[\pm 45/0/90]_S$ laminate is made of composite layers with lamina constants $E_1 = 20 \times 10^6$ psi, $E_2 = 1.5 \times 10^6$ psi, $G_{12} = 10^6$ psi, $\nu_{12} = 0.29, X = X' = 310$ ksi, Y=9 ksi, $Y' = 30$ ksi, and S=15 ksi. The thickness of each layer is 0.005”. This laminate is subject to N_{12}. Assume that the composite material fails according to the

Tsai-Wu failure criterion. Compute the maximum allowable N_{12} before any of the layers fail. Which layer fails first?

Solution: First, we need to evaluate the plate stiffness of the laminate. Because it is a symmetric laminate loaded only by the in-plane load N_{12}, the extension stiffness matrix A is sufficient for our analysis. As we have learned previously, we can obtain plane-stress reduced stiffness matrix Q for the composite lamina, then transform Q according to the layup orientation, and integrate the transformed Q matrices through the thickness to obtain A matrix as

$$A = \begin{bmatrix} 348925 & 101315 & 0 \\ 101315 & 348925 & 0 \\ 0 & 0 & 123805 \end{bmatrix} \text{lb/in}$$

We can obtain the in-plane plate strains due to the applied load N_{12} as

$$\epsilon = A^{-1} \begin{Bmatrix} 0 \\ 0 \\ N_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 8.07723 \times 10^{-6} \end{Bmatrix} N_{12}\text{in/lb}$$

Since $\kappa = 0$, the 3D in-plane strains are $\varepsilon_e = \epsilon$. Next, we need to compute the 3D stresses for each layer. We need to first use $\sigma'_e = Q'\varepsilon_e$ to compute 3D stresses in the laminate coordinate system, then we need to transform the 3D stresses into the material coordinate system using $\sigma_e = R_{\sigma e}^{-1}\sigma'_e$. Following this procedure, we obtain

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{\pm 45} = \begin{Bmatrix} \pm 79.517 \\ \mp 4.328 \\ 0 \end{Bmatrix} N_{12}/\text{in}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{0} = \begin{Bmatrix} 0 \\ 0 \\ 8.077 \end{Bmatrix} N_{12}/\text{in}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{90} = \begin{Bmatrix} 0 \\ 0 \\ -8.077 \end{Bmatrix} N_{12}/\text{in}$$

Next, we need to plug the 3D stresses into the Tsai-Wu failure criterion for each layer, to compute the maximum allowable N_{12}. For the $\pm 45^\circ$ layers, according to Eq. (6.49), we have

$$\left(\frac{1}{Y} - \frac{1}{Y'}\right)(\mp 4.328 N_{12}) + \frac{(\pm 79.517 N_{12})^2}{XX'} + \frac{(\mp 4.328 N_{12})^2}{YY'} + \frac{79.517 \times 4.328 N_{12}^2}{X^2} = 1$$

which can be solved as $N_{12} = \mp 1732.78$ lb/in and $N_{12} = \pm 4158.82$ lb/in. This implies that the 45 layer will fail when $N_{12} = -1732.78$ lb/in or $N_{12} = 4158.82$ lb/in and the -45° layer will fail when $N_{12} = 1732.78$ lb/in or $N_{12} = -4158.82$ lb/in. It is interesting to note that these two layers fail at different magnitudes of in-plane shear force along the positive shear direction or negative shear direction.

For the 0° layers, according to Eq. (6.49), we have

$$\left(\frac{8.077 N_{12}}{S}\right)^2 = 1$$

which can be solved as $N_{12} = \pm 1857.07$ lb/in.

For the $90°$ layers, according to Eq. (6.49), we have

$$\left(\frac{-8.077N_{12}}{S}\right)^2 = 1$$

which can be solved as $N_{12} = \pm 1857.07$ lb/in. When we subject the laminate to a positive shear force N_{12}, the $-45°$ layer will fail first at $N_{12} = 1732.78$ lb/in. When we subject the laminate to a negative shear force N_{12}, the $45°$ layer will fail first at $N_{12} = -1732.78$ lb/in.

6.3.4 Hashin failure criterion

The Tsai-Wu failure criterion represents an improvement over the Tsai-Hill failure criterion by considering the possibility of different compressive and tensile strengths. However, both failure criteria share the same disadvantage that they can not clearly indicate the failure modes. Particularly for unidirectional fiber reinforced composites, we know that they could fail due to multiple, drastically different failure mechanisms such as fiber breakage in tension, fiber buckling in compression, or matrix cracking in tension. It is not clear that all these physically distinct failure modes can be governed by a single smooth failure function given by the Tsai-Wu failure criterion or the Tsai-Hill failure criterion. Hashin [64] proposed a failure criterion for unidirectional fiber composites which can take distinct failure modes into consideration. It is assumed that the unidirectional fiber composite material is a transversely isotropic homogeneous material and will fail in four different modes including tensile and compressive fiber modes, and tensile and compressive matrix modes. The strength of such a material can be characterized using the tensile strength (X) and compressive strength (X') along the fiber direction, tensile strength (Y) and compressive strength (Y') along the matrix direction (x_2 or x_3 direction), transverse shear strength R in $x_2 - x_3$ plane, and axial shear strength S in $x_1 - x_2$ plane and $x_1 - x_3$ plane.

Instead of using stress components, Hashin proposed to use stress invariants for the transversely isotropic material with x_1 along the fiber direction and x_2, x_3 in the plane perpendicular to the fiber. For any arbitrary rotation around the fiber direction, we will have the following four distinct stress invariants from linear up to the quadratic terms of stress components.

$$I_1 = \sigma_{11}, \qquad I_2 = \sigma_{22} + \sigma_{33}, \qquad I_3 = \sigma_{23}^2 - \sigma_{22}\sigma_{33}, \qquad I_4 = \sigma_{12}^2 + \sigma_{13}^2 \tag{6.50}$$

Thus, a general quadratic failure criterion can be expressed in terms of these stress invariants as

$$f = A_1 I_1 + B_1 I_1^2 + A_2 I_2 + B_2 I_2^2 + C_{12} I_1 I_2 + A_3 I_3 + A_4 I_4 = 1 \tag{6.51}$$

This failure criterion can be easily calibrated using the shear strength in $x_2 - x_3$ plane R ($\sigma_{23} = R$ and all other components vanish) and shear strength in $x_1 - x_2$ plane or $x_1 - x_3$ plane S ($\sigma_{12} = S$ or $\sigma_{13} = S$ and all other components vanish) as

$$A_3 R^2 = 1, \qquad A_4 S^2 = 1 \tag{6.52}$$

Thus, we have

$$A_3 = \frac{1}{R^2}, \qquad A_4 = \frac{1}{S^2} \tag{6.53}$$

For the tensile fiber mode ($\sigma_{11} > 0$), Hashin assumed that $\sigma_{11}, \sigma_{12}, \sigma_{13}$ contribute to this failure mode and the contributions from other stress components can be neglected (i.e. we can assume $I_2 = I_3 = 0$), thus the failure function in Eq. (6.51) can be written as

$$f = A_1\sigma_{11} + B_1\sigma_{11}^2 + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.54}$$

Since all the calibration information we have about this tensile fiber failure mode is $\sigma_{11} = X$, it is impossible to determine both A_1 and B_1. Instead, Hashin assumed the contribution of the linear term due to A_1 can be neglected and approximately calibrated the failure criterion as

$$f = \frac{\sigma_{11}^2}{X^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.55}$$

For compressive fiber mode ($\sigma_{11} < 0$), because of the complexity of this failure mechanism and lack of evidence that axial shear stresses contribute to the compressive fiber failure, Hashin proposed to approximate this failure mode using the simple maximum normal stress failure criterion:

$$f = \frac{|\sigma_{11}|}{X'} = 1 \tag{6.56}$$

It is noted that Hashin purposely tried to separate the tensile fiber mode from the compressive fiber mode, thus he chose not to use X and X' together to determine A_1 and B_1 similar to what has been done for the Tsai-Wu failure criterion.

For the matrix failure, Hashin argued that only σ_{11} does not contribute to this failure mode. Thus, the failure criterion can be written as

$$f = A_2(\sigma_{22} + \sigma_{33}) + B_2(\sigma_{22} + \sigma_{33})^2 + \frac{\sigma_{23}^2 - \sigma_{22}\sigma_{33}}{R^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.57}$$

For the tensile matrix failure, Hashin introduced the same approximation as fiber tensile failure (i.e., neglecting the contribution from the linear term) such that

$$f = \frac{(\sigma_{22} + \sigma_{33})^2}{Y^2} + \frac{\sigma_{23}^2 - \sigma_{22}\sigma_{33}}{R^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.58}$$

For the compressive matrix failure, we can first calibrate the criterion using the compressive strength Y' as

$$f = -A_2Y' + B_2Y'^2 = 1 \tag{6.59}$$

Next, Hashin argued that matrix can fail under bi-axial compressive pressure so that $\sigma_{22} = \sigma_{33} = -\sigma$. Substituting this stress state into Eq. (6.57), we have

$$f = -2A_2\sigma + 4B_2\sigma^2 - \frac{\sigma^2}{R^2} = 1 \tag{6.60}$$

We can solve A_2 and B_2 from Eqs. (6.59) and (6.60) as

$$A_2 = \frac{Y'^2\left(\sigma^2/R^2 + 1\right) - 4\sigma^2}{4\sigma^2Y' - 2\sigma Y'^2}, \qquad B_2 = \frac{2R^2\sigma - R^2Y' - \sigma^2Y'}{2R^2\sigma Y'(Y' - 2\sigma)} \tag{6.61}$$

It is reasonable to expect that $\sigma \gg Y'$ because it will be much more difficult for the matrix material to fail under equal bi-axial pressure. If we only keep the leading terms in A_2 and B_2 in terms of Y'/σ, we have

$$A_2 = \frac{Y'}{4R^2} - \frac{1}{Y'}, \qquad B_2 = \frac{1}{4R^2} \tag{6.62}$$

Thus, the resulting failure criterion for the compressive matrix mode is

$$f = \left[\left(\frac{Y'}{2R}\right)^2 - 1\right]\frac{\sigma_{22} + \sigma_{33}}{Y'} + \left(\frac{\sigma_{22} + \sigma_{33}}{2R}\right)^2 + \frac{\sigma_{23}^2 - \sigma_{22}\sigma_{33}}{R^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.63}$$

One complexity is to determine when we consider that the matrix is under compression or tension. Hashin suggested that when $\sigma_{22} + \sigma_{33} \geq 0$ the material is in the tensile matrix mode. Otherwise, it is in the compressive matrix mode.

In summary, for unidirectional fiber reinforced composites, after obtaining the stresses in the material coordinates, Hashin suggests the following failure criterion for four possible distinct failure modes

- Tensile fiber mode ($\sigma_{11} \geq 0$)

$$f = \frac{\sigma_{11}^2}{X^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.64}$$

- Compressive fiber mode ($\sigma_{11} < 0$)

$$f = \frac{|\sigma_{11}|}{X'} = 1 \tag{6.65}$$

- Tensile matrix mode ($\sigma_{22} + \sigma_{33} \geq 0$)

$$f = \frac{(\sigma_{22} + \sigma_{33})^2}{Y^2} + \frac{\sigma_{23}^2 - \sigma_{22}\sigma_{33}}{R^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.66}$$

- Compressive matrix mode ($\sigma_{22} + \sigma_{33} < 0$)

$$f = \left[\left(\frac{Y'}{2R}\right)^2 - 1\right]\frac{\sigma_{22} + \sigma_{33}}{Y'} + \left(\frac{\sigma_{22} + \sigma_{33}}{2R}\right)^2 + \frac{\sigma_{23}^2 - \sigma_{22}\sigma_{33}}{R^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} = 1 \tag{6.67}$$

It is noted that the material could fail in both fiber mode and matrix mode, thus both types of failure modes should be checked at the same time for the same material point.

For a plane-stress state, we know $\sigma_{i3} = 0$. The Hashin failure criteria is simplified to be

- Tensile fiber mode ($\sigma_{11} \geq 0$)

$$f = \left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.68}$$

- Compressive Fiber mode ($\sigma_{11} < 0$)

$$f = \frac{|\sigma_{11}|}{X'} = 1 \tag{6.69}$$

- Tensile matrix mode ($\sigma_{22} \geq 0$)

$$f = \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.70}$$

- Compressive matrix mode ($\sigma_{22} < 0$)

$$f = \left[\left(\frac{Y'}{2R}\right)^2 - 1\right]\frac{\sigma_{22}}{Y'} + \left(\frac{\sigma_{22}}{2R}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.71}$$

EXAMPLE 6.9

Strength analysis of a composite lamina: Suppose that a composite lamina can be assumed as a homogeneous material. The composite material has strengths X=100 ksi, $X' = 30$ ksi, Y=9 ksi, $Y' = 20$ ksi and S=15 ksi. An off-axis lamina is made of this material and loaded by a tensile σ_0 as shown in Figure 6.10. Predict the maximum allowable σ_0 as a function of the fiber orientation angle θ_3 according to the Tsai-Wu failure criterion and the Hashin failure criterion.

Solution: First, we need to evaluate the stress components in the material coordinate system (x_i') due to σ_0 in the problem coordinate system x_i. We have

$$\begin{aligned} \sigma_{11} &= \cos^2\theta_3\sigma_0 \\ \sigma_{22} &= \sin^2\theta_3\sigma_0 \\ \sigma_{12} &= -\cos\theta_3\sin\theta_3\sigma_0 \end{aligned}$$

According to the Tsai-Wu failure criterion in Eq. (6.49), we have

$$\left(\frac{1}{X}-\frac{1}{X'}\right)\cos^2\theta_3\sigma_0+\left(\frac{1}{Y}-\frac{1}{Y'}\right)\sin^2\theta_3\sigma_0+\frac{\left(\cos^2\theta_3\sigma_0\right)^2}{XX'}+$$
$$\frac{\left(\sin^2\theta_3\sigma_0\right)^2}{YY'}-\frac{\left(\cos\theta_3\sin\theta_3\sigma_0\right)^2}{XX'}+\left(\frac{\cos\theta_3\sin\theta_3\sigma_0}{S}\right)^2=1$$

This equality has two solutions which can be obtained easily by a symbolic manipulator. The lengthy formulas are not given here for the sake of saving space.

Under this particular loading, we have σ_{11} and σ_{22} always greater than zero. Thus, we could have possible tensile fiber mode and tensile matrix mode according to the Hashin failure criterion. For tensile fiber mode, according to Eq. (6.68), we have

$$\left(\frac{\cos^2\theta_3\sigma_0}{X}\right)^2+\left(\frac{\cos\theta_3\sin\theta_3\sigma_0}{S}\right)^2=1$$

For tensile matrix mode, according to Eq. (6.70), we have

$$\left(\frac{\sin^2\theta_3\sigma_0}{Y}\right)^2+\left(\frac{\cos\theta_3\sin\theta_3\sigma_0}{S}\right)^2=1$$

Both equations can be solved symbolically. It can be found out that when θ_3 increases from 0 to $16.6992°$, σ_0 solved from the first equation is always smaller than that solved from the second equation, which implies that the lamina failures in tensile fiber mode. When θ_3 increases from $16.6992°$ to $90°$, σ_0 solved from the first equation is always larger than that solved from the second equation, which implies that the lamina failures in tensile matrix mode. The changes of maximum σ_0 with respect to θ_3 for both failure criteria are plotted in Figure 6.12. Both failure criteria predict similar trends. These two predictions have noticeable differences for off-axis angles of a few degrees to about $45°$. It is also noted that Tsai-Wu represents a smooth curve while Hashin has a discontinuity at $\theta_3 = 16.6992°$ due to change of failure modes.

6.4 Strength ratio

For a structure subject to a load P, we can solve for the 3D stress field, then we can compute the failure index at each point according to a chosen failure criterion. Failure index is a

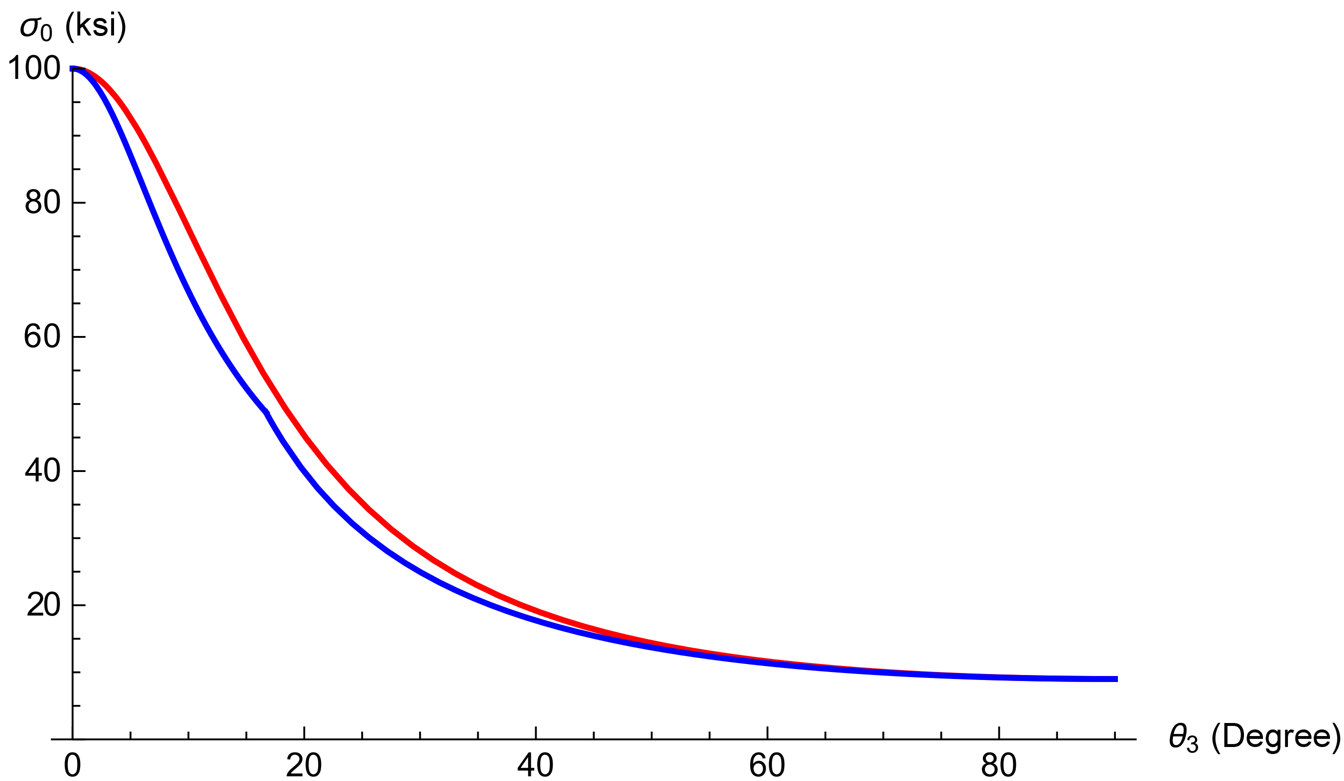

Figure 6.12 Strength of an off-axis lamina predicted by different failure criteria (Red - Tsai-Wu criterion; Blue - Hashin criterion).

pointwise quantity varying within the structure. If any of the failure indexes is greater or equal to 1, the structure fails at the corresponding point. If all the failure indices are smaller than 1, the structure is safe. Sometimes, f is computed to be smaller than zero according to some failure criteria such as the Tsai-Wu failure criterion. It means that according to this particular failure criterion, material point is safer under the corresponding stress state than a stress state which could result in $f = 0$, which does not make physical sense. Since a negative f is not physically meaningful, we will replace all negative f values with zero. The initial failure load of a structure, P_{cr}, is defined as the load under which the maximum failure index is equal to 1. P_{cr} is called *the first point failure load* or *the initial failure load* of the structure. If the maximum failure index is smaller than 1, we can increase the load until it reaches the value P_{cr} so that the maximum failure index is equal to 1. If the maximum failure index is greater than 1, we can decrease the load until it reaches the value P_{cr} so that the maximum failure index is equal to 1.

If the stress analysis is linear, then instead of continuously increasing or decreasing the load to carry out the stress analysis multiple times, we only need one analysis. For an arbitrary load P, there is a corresponding 3D stress field σ_{ij}. We can compute the failure index for each point based on this stress field. Suppose that the initial failure load is $P_{cr} = \alpha P$, then the corresponding stress field is $\alpha\sigma_{ij}$, with α being positive because we are predicting the failure load in the direction of load P.

If the failure criterion is linear with respect to the stress field, such as the maximum stress failure criterion, we will have

$$f(\alpha\sigma_{ij}) = \alpha f(\sigma_{ij})$$

According to the failure criterion, we require $\alpha f = 1$. Thus, we have

$$\alpha = \frac{1}{f} \tag{6.72}$$

α is also commonly called *the strength ratio*. We can compute the strength ratio at each point, which implies how many times of the current load it will take to fail that point. For example, if the strength ratio computed at one point is 2.5, it means that we need to increase the current load to be 2.5 times larger in the same direction to fail the material at this point. The strength ratio is also sometimes called the safety margin. It is emphatically pointed out that the simple reciprocal relation between the strength ration and the failure index in Eq. (6.72) only holds for linear failure criteria. For other failure criteria which are not linear, it is more intuitive to use the strength ratio which will be illustrated using an example later.

Denote the smallest α among all the points as α_{min}, the initial failure load can be computed as

$$P_{cr} = \alpha_{min} P \tag{6.73}$$

If we let P equal to 1, then α_{min} is the initial failure load.

If the failure criterion contains only quadratic terms of the stress components such as the Mises failure criterion and the Tsai-Hill failure criterion. We will have $f(\alpha\sigma_{ij}) = \alpha^2 f(\sigma_{ij})$. According to the failure criterion, we require $\alpha^2 f = 1$. Thus, $\alpha = \frac{1}{\sqrt{f}}$. If f is smaller or equal to zero, $\alpha = +\infty$ which means that the corresponding material point will not fail no matter how large the load is.

If the failure criterion contains both linear terms and quadratic terms of the stress components such as the Tsai-Wu failure criterion, we will have

$$f(\alpha\sigma_{ij}) = \alpha^2 a + \alpha b = 1 \tag{6.74}$$

with

$$a = F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + F_{33}\sigma_{33}^2 + 2F_{12}\sigma_{11}\sigma_{22} + 2F_{13}\sigma_{11}\sigma_{33} + 2F_{23}\sigma_{22}\sigma_{33} + F_{44}\sigma_{23}^2 + F_{55}\sigma_{13}^2 + F_{66}\sigma_{12}^2 \tag{6.75}$$

and

$$b = F_1\sigma_{11} + F_2\sigma_{22} + F_3\sigma_{33} \tag{6.76}$$

where a, b are computed based on the stress state due to load P. Then, we can solve α as

$$\alpha = \frac{-b \pm \sqrt{b^2 + 4a}}{2a} \tag{6.77}$$

Only the positive solution makes sense. If a is positive, we have

$$\alpha = \frac{-b + \sqrt{b^2 + 4a}}{2a} \tag{6.78}$$

If a is negative, a positive α exists only if $b > 0$, the smaller value of the two possible solutions is the same as Eq. (6.78). If a and b are both negative, a positive α does not exist. If a positive solution does not exist, for example $\alpha < 0$ or $b^2 + 4a < 0$, it means that it is impossible to achieve $f = 1$, which further means it is impossible to fail the material point and thus $\alpha = +\infty$.

If we also want to find the initial failure load along the negative direction of load P, we need to flip the sign of the stress results we have obtained under load P and apply the failure criterion to compute the strength ratio again. However, if the failure function remains the same for both σ_{ij} and $-\sigma_{ij}$ such as the Mises failure criteria, the maximum shear stress failure criterion, and the Tsai-Hill failure criterion, the strength ratio α will remain the same. For the Tsai-Wu failure criterion, to compute the initial failure load along the negative direction of P, we need to switch the sign of b to be $b^* = -b$. Then the failure criterion is written as

$$f(\alpha\sigma_{ij}) = \alpha^2 a + \alpha b^* = 1 \tag{6.79}$$

if $a > 0$

$$\alpha = \frac{-b^* + \sqrt{b^{*2} + 4a}}{2a} \tag{6.80}$$

If $a < 0$, a positive α exists only if $b^* > 0$ (or $b < 0$), the smaller value of the two possible solutions is the same as the above equation. If $a < 0$ and $b^* < 0$ (or $b > 0$), a positive α does not exist.

Composite materials usually have residual stresses in the material before some external loads are applied. For this situation, the simplicity of computing the initial failure load using just one stress analysis does not exist for most failure criteria. Suppose a residual stress field σ_{ij}^0 exists in the material before a load P is applied to the structure. Suppose that the stress field σ_{ij} is generated due to P, then the total stress field is $\sigma_{ij} + \sigma_{ij}^0$. If the initial failure load $P_{cr} = \alpha P$ is applied, $\alpha\sigma_{ij}$ will be generated and the total stress field is $\alpha\sigma_{ij} + \sigma_{ij}^0$.

It is not straightforward to compute the strength ratio in a simple way by using only one stress analysis. Usually, special consideration needs to be given for a specific failure criterion. If the failure criterion is governed by a single formula, computing the initial failure load using just one stress analysis is still possible. For example, for the Tsai-Wu failure criterion, we have

$$f(\sigma_{ij}^0 + \alpha\sigma_{ij}) = f(\sigma_{ij}^0) + \alpha^2 a + \alpha c = 1 \tag{6.81}$$

with

$$\begin{aligned} c =& b + 2\left[F_{11}\sigma_{11}^0\sigma_{11} + F_{22}\sigma_{22}^0\sigma_{22} + F_{33}\sigma_{33}^0\sigma_{33} + \right. \\ & F_{12}(\sigma_{11}^0\sigma_{22} + \sigma_{22}^0\sigma_{11}) + F_{13}(\sigma_{11}^0\sigma_{33} + \sigma_{33}^0\sigma_{11}) + F_{23}(\sigma_{22}^0\sigma_{33} + \sigma_{33}^0\sigma_{22}) \\ & \left. + F_{44}\sigma_{23}^0\sigma_{23} + F_{55}\sigma_{13}^0\sigma_{13} + F_{66}\sigma_{12}^0\sigma_{12}\right] \end{aligned} \tag{6.82}$$

and a, b given in Eq. (6.75) and Eq. (6.76). If we assume that the material has not failed under the initial stress (i.e., $f(\sigma_{ij})^0 < 1$), the strength ratio along the direction of P can be computed as

$$\alpha = \frac{-c + \sqrt{c^2 + 4a[1 - f(\sigma_{ij}^0)]}}{2a} \tag{6.83}$$

To compute the initial failure load along the negative direction of P, we need to switch the sign of c, the strength ratio can be computed as

$$\alpha = \frac{c + \sqrt{c^2 + 4a[1 - f(\sigma_{ij}^0)]}}{2a} \tag{6.84}$$

The strength ratio and the initial failure load in terms of strains can be calculated in the same way if the failure criterion can be conveniently expressed in terms of strains. If more than one failure functions are used, such as the Hashin criterion, one needs to check both fiber and matrix failure at a point, then the failure functions of all the applicable failure modes should be evaluated and the largest failure index (or smallest strength ratio) should be used as the failure index (or strength ratio) for that point.

EXAMPLE 6.10

A composite material has strength constants as $X = 1168$ MPa, $X' = 740$ MPa, $Y = Z = Y' = Z' = 99$ MPa, $R = 450$ MPa, $T = S = 68.6$ MPa. The stress state of a point in a composite laminate is computed as $\sigma_{11} = 42.88$ MPa, $\sigma_{22} = 5.33$ MPa, $\sigma_{33} = 14.68$ MPa, $\sigma_{23} = 1.6$ MPa, $\sigma_{13} = 0.5$ MPa, $\sigma_{12} = 0$ MPa. Evaluate the failure index f and strength ratio α according to the Tsai-Wu failure criterion and the Hashin failure criterion.

Solution: First, let us use the Tsai-Wu failure criterion. Based on the given strength constants of the material, we can compute the parameters needed for the Tsai-Wu failure criterion according to Eq. (6.39), (6.40), (6.41), and (6.47) as:

$$\begin{aligned} &F_1 = -4.95 \times 10^{-4}\text{MPa}^{-1}, \qquad F_2 = F_3 = 0\text{MPa}^{-1} \\ &F_{11} = 1.157 \times 10^{-6}\text{MPa}^{-2}, \qquad F_{22} = F_{33} = 1.0203 \times 10^{-4}\ \text{MPa}^{-2} \\ &F_{44} = -4.94 \times 10^{-6}\ \text{MPa}^{-2}, \qquad F_{55} = F_{66} = 2.125^{-4}\text{MPa}^{-2} \\ &2F_{23} = -2.029 \times 10^{-4}\text{MPa}^{-2}, \qquad 2F_{12} = 2F_{13} = -1.157 \times 10^{-6}\text{MPa}^{-2} \end{aligned}$$

Then, we can compute $a = 0.0102107$ and $b = -0.0212336$. Thus, the failure index

$$f = a + b = -0.011023$$

Since the failure index is negative, we can effectively replace it with $f = 0$ as if this stress state has no effect on the damage of the material at all. However, it does not imply that the strength ratio is infinite, because we can compute the strength ratio according to Eq. (6.78) as

$$\alpha = 10.99$$

Indeed, if the load is increased $\alpha = 10.99$ times, we will have the stress state become $\sigma_{11} = 471.38$ MPa, $\sigma_{22} = 58.58$ MPa, $\sigma_{33} = 161.34$ MPa, $\sigma_{23} = 17.58$ MPa, $\sigma_{13} = 5.50$ MPa, $\sigma_{12} = 0$ MPa. Under this stress state, we have $a = 1.233$ and $b = -0.233$ and $f = 1.0$, which implies that the material fails.

Now, let us use the Hashin failure criterion. For the given stress state, because $\sigma_{11} \geq 0$ and $\sigma_{22} + \sigma_{33} \geq 0$, the material could fail either in the tensile fiber mode or the tensile matrix mode. According to Eqs. (6.64), we have $f = 0.0014$ for the fiber tensile mode. The corresponding strength ratio is $\alpha = 1\sqrt{f} = 26.72$. According to Eq. (6.66), we have $f = 0.0405$ for the matrix tensile mode. The corresponding strength ratio is $\alpha = 1\sqrt{f} = 4.967$. The larger failure index ($f = 0.0405$) among these two failure indexes will be the failure index for this material and the corresponding mode is tensile matrix mode. The smaller strength ratio ($\alpha = 4.967$) among the two strength ratios will be the corresponding strength ratio for this material and the corresponding failure mode remains the same as the tensile matrix mode.

Clearly, this example demonstrates that there are no direct relations between the failure index and the strength ratio and it is more meaningful to use the strength ratio as the indicator for the safety margin of the material.

It is also shown that different failure criteria could predict very different failure indexes or strength ratios for the same material. Experimental data should be used to decide which failure criterion provides a better prediction for a certain material.

6.5 Failure envelope

Failure envelope graphically depicts the boundary of the stress/strain states so that the material fails outside the boundary and the material is safe inside the boundary. In general, it could be a six-dimensional surface in terms of the six stress components or six strain components, or a mix of stress and strain components. However, it is very difficult to graphically represent and visualize such surfaces. Although we can plot such envelopes in the three-dimensional space of principal stresses for isotropic materials, we do not plot such envelopes for general anisotropic or orthotropic materials because the failure criteria are not expressed in terms of the principal stresses. Thus, normally, the failure envelopes of composite materials are generated for bi-axial or tri-axial loading conditions. If the stress analysis and failure analysis can be performed analytically, the failure envelope can be evaluated relatively easily, as shown in previous examples. However, if the stress analysis and failure analysis are performed using numerical methods, it becomes more involved. Usually, we hold one stress σ_{11} to be constant, then change the another stress, say σ_{22}, until the material fails and $(\sigma_{11}, \sigma_{22})$ will become a data point on the failure envelope. Usually, for one stress σ_{11}, there are more than one σ_{22} values which will fail the material. Next, increase σ_{11} and find another set of σ_{22} to locate the next data point on the failure envelope. For a linear stress analysis, the search for σ_{22} can be simplified as in the case of the material with residual stresses which was discussed above.

The same concept can also be applied to obtain the failure envelope of a composite laminate. Generally speaking, the failure envelope of a composite laminate can be a six-