Numerical characterization of effective fully coupled thermo-electro-magneto-viscoelastic-plastic response of smart composites

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**Abstract**

The focus of the present paper is to construct a general purpose micromechanics model to predict the effective fully coupled time-dependent and non-linear multiphysics responses of smart composites. The present model is established on the basis of the variational asymptotic method and implemented using the finite element method. In light of the time-dependent and non-linear characteristics of composites, an incremental procedure in conjunction with an instantaneous tangential electromagneto-mechanical matrix of composites was established. The accuracy of the proposed model was verified through the comparison with ABAQUS results. Finally, a numerical example was employed to demonstrate the capability of the proposed model.

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**1. Introduction**

The smart composite consisting of piezoelectric and piezomagnetic constituents displays a magneto-electric coupling effect that is absent in constituents [1–9]. The magneto-electric coupling effect created by the interaction of piezoelectric phases and piezomagnetic phases has recently been extensively investigated due to their broad engineering applications [10–12]. Since the piezoelectric and piezomagnetic ceramics are brittle and susceptible to fracture, adding a polymer or metallic alloy matrix into the two-phase electromagnetoelastic composite will increase the ductility and formability of the composites. To date, several investigations have been conducted for the response of smart composites containing metallic phases. For example, Bednarcyk [13] developed a micro-macro theory to predict the fully coupled electro-magneto-thermo-elasto-plastic behavior of arbitrary composite laminates using Generalized Method of Cell (GMC). Due to the introduction of linear viscoelastic polymer matrix, the composites exhibit time dependent behavior [14]. The reports that are involved in the response of smart composites containing both metallic phases and viscoelastic phases is still limited. Therefore, there is a need to develop an efficient micromechanical tool for the analysis and design of such composites.

The goal of this paper is to develop a general purpose micromechanics model for predicting the time-dependent, non-linear, and multiphysics response of smart composites. In light of the time-dependent characteristics and non-linearity of constitutive relations, an incremental procedure associated with instantaneous tangential electromagneto-mechanical matrix was established based on the micromechanics framework VAMUCH [15]. In order to demonstrate the capability, a smart composites consisting of metallic phase, piezoelectric material, piezomagnetic material, and linear viscoelastic matrix was analyzed using the proposed model.

**2. Incremental constitutive equations of materials**

**2.1. Constitutive equations for linear thermo-viscoelastic polymer**

Considering the linear thermo-viscoelastic polymer having no history of stress and deformation before time $t=0$, then based on the Boltzmann superposition principle, the constitutive equations for the linear thermo-viscoelastic polymer can be expressed in the time domain in the following way:

\[
\sigma_{ij}(t) = \int_0^t \left[ B_{ijkl}(t-\tau)\varepsilon_{kl}(\tau) + \beta_{ijkl}(t-\tau)\theta(\tau) \right] d\tau
\]

\[
D_i(t) = \int_0^t \left[ k_{ij}(t-\tau)\dot{E}_j(\tau) + p_i(t-\tau)\dot{\theta}(\tau) \right] d\tau
\]
where $B_{ijkl}(t)$, $k_{ij}(t)$, and $\mu_{ij}(t)$ are the stress relaxation stiffness, dielectric tensor, and magnetic permeability tensor, respectively; $\varepsilon_{il}(t)$ is the strain rate; and $E_{i}(t)$ and $H_{j}(t)$ are the electric field rate and magnetic field rate, respectively; $\theta(t)$ is the temperature change rate; $\sigma_{ij}(t)$, $D_{i}(t)$, and $B_{i}(t)$ are the instantaneous stress tensor, electrical displacement vector, and magnetic induction vector, respectively; $\beta_{ij}(t)$, $p_{i}(t)$, and $m_{i}(t)$ are the instantaneous thermal stress tensor, pyroelectric vector, and pyromagnetic vector, respectively. Note that $\beta_{ij}(t) = -B_{ijkl}(t)\alpha_{ij}$ with $\alpha_{ij}$ being thermal expansion coefficients. In this study, the $\alpha_{ij}$ is assumed to be constant.

According to the time–temperature superposition principle [16], the real time $t$ has to be replaced with reduced time $\xi$ in order to account for the variation of material’s properties of polymer with temperature. Hence, Eq. (1a)–(1c) can be rewritten as

$$\sigma_{ij}(t) = \int_{0}^{t} \left[ B_{ijkl}(\xi - \xi') \varepsilon_{kl}(\xi') + \beta_{ij}(\xi - \xi') \theta(\xi') \right] d\xi'$$

(2a)

$$D_{i}(t) = \int_{0}^{t} \left[ k_{ij}(\xi - \xi') E_{j}(\xi') + p_{i}(\xi - \xi') \theta(\xi') \right] d\xi'$$

(2b)

$$B_{i}(t) = \int_{0}^{t} \left[ \mu_{ij}(\xi - \xi') H_{j}(\xi') + m_{i}(\xi - \xi') \theta(\xi') \right] d\xi'$$

(2c)

The reduced time $\xi = \xi(t)$ is defined by

$$\xi(t) = \int_{0}^{t} \frac{dt'}{\alpha_{j}}$$

(3)

where $\alpha_{j}$ is a time-scale shift factor, and $\xi' = \xi(t)$.

As pointed out by Pyatigorets et al. [17], since the corresponding value of real time $t$ can be found for each value of reduced time $\xi$ and vice versa, the stress and strain in the reduced time domain can be replaced with their values found for the corresponding real time, such that

$$\sigma_{ij}(\xi) = \sigma_{ij}(\xi(t)) = \xi_{ij}(\xi(t)) \equiv \xi_{ij}(t)$$

(4)

Hence, the Eq. (2a)–(2c) can be simplified as

$$\sigma_{ij}(t) = \int_{0}^{t} \left[ B_{ijkl}(t - \xi(t)) \varepsilon_{kl}(t) + \beta_{ij}(t - \xi(t)) \theta(t) \right] dt$$

(5a)

$$D_{i}(t) = \int_{0}^{t} \left[ k_{ij}(t - \xi(t)) E_{j}(t) + p_{i}(t - \xi(t)) \theta(t) \right] dt$$

(5b)

$$B_{i}(t) = \int_{0}^{t} \left[ \mu_{ij}(t - \xi(t)) H_{j}(t) + m_{i}(t - \xi(t)) \theta(t) \right] dt$$

(5c)

In light of the non-linear, time dependent, and multiphysics response of the composites, our analysis need to be incremental. The incremental formulations of Eq. (5a)–(5c) can be expressed as

$$\Delta \sigma_{ij}(t) = \sigma_{ij}(t + \Delta t) - \sigma_{ij}(t)$$

$$\Delta D_{i}(t) = D_{i}(t + \Delta t) - D_{i}(t)$$

$$\Delta B_{i}(t) = B_{i}(t + \Delta t) - B_{i}(t)$$

(6a)

(6b)

(6c)

Although the strain rate, electrical field rate, and magnetic field rate are not necessarily constant in the whole time domain, it is reasonable to assume that the strain rate, electrical field rate, and magnetic field rate are kept constant during each time increment $\Delta t$. The temperature change rate can be kept uniform in the whole composites. Then, the Eq. (6a)–(6c) can be rephrased as

$$\Delta \sigma_{ij}(t) = L_{ijkl}(t)\Delta \varepsilon_{kl}(t) + \gamma_{ij}(t) \Delta \theta(t) + o_{ij}(t)$$

(7a)

with

$$L_{ijkl}(t) = \frac{1}{\Delta t} \int_{t}^{t + \Delta t} B_{ijkl}(\xi(t + \Delta t) - \xi(t)) d\xi$$

$$\gamma_{ij}(t) = \frac{1}{\Delta t} \int_{t}^{t + \Delta t} [\beta_{ij}(\xi(t + \Delta t) - \xi(t))] dt$$

$$o_{ij}(t) = \int_{0}^{t} \left[ B_{ijkl}(\xi(t + \Delta t) - \xi(t)) - B_{ijkl}(\xi(t) - \xi(t))] \varepsilon_{kl}(t) d\xi + \int_{0}^{t} \left[ p_{ij}(\xi(t + \Delta t) - \xi(t)) - p_{ij}(\xi(t) - \xi(t))] \theta(t) d\xi - \Delta \theta(t) = -K_{ijkl}(t) \Delta \theta(t) - M_{ij}(t) \Delta \theta - \psi_{ij}(t)$$

(7b)

with

$$K_{ijkl}(t) = \frac{1}{\Delta t} \int_{t}^{t + \Delta t} k_{ij}(\xi(t + \Delta t)) - \xi(t)) d\xi$$

$$M_{ij}(t) = \frac{1}{\Delta t} \int_{t}^{t + \Delta t} m_{ij}(\xi(t + \Delta t)) - \xi(t)) d\xi$$

$$\psi_{ij}(t) = \int_{0}^{t} \left[ \mu_{ij}(\xi(t + \Delta t)) - \xi(t)) - \mu_{ij}(\xi(t) - \xi(t))] H_{j}(t) d\xi + \int_{0}^{t} \left[ m_{ij}(\xi(t + \Delta t)) - \xi(t)) - m_{ij}(\xi(t) - \xi(t))] \theta(t) d\xi$$

(7c)

2.2. Constitutive equations for piezoelectric-piezomagnetic materials

The elastic and the dielectric responses are coupled in piezoelectric materials where the mechanical variables of stress, strain are related to each other as well as to the electric variables of electric field and electric displacement. The coupling between mechanical and electric fields is described by piezoelectric coefficients. The linear rate independent coupled constitutive equations of piezoelectric materials are given by

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} - e_{ik} E_{k} - q_{ij} H_{k} + \beta_{ij} \theta$$

(8a)

$$D_{i} = e_{ik} \varepsilon_{kl} + q_{ij} H_{k} + a_{k} H_{k} + p_{i} \theta$$

(8b)
\[
\begin{align*}
B_i &= e_{ijkl} x_{ij} q_{ik} + a_{ik} H_k + \mu_{ik} H_k + m_{ik} \\
\text{where } C_{ijkl}, e_{ijkl}, q_{ijkl}, \text{ and } \beta_{ij} \text{ are the elastic, piezoelectric, piezomagnetic and thermal stress tensors, respectively (note that } \\
\beta_{ij} &= -C_{ijkl} a_{ij} \text{ with } a_{ij} \text{ as the thermal expansion strain tensor); } \\
k_{ij}, a_{ik}, \text{ and } \mu_{ik} \text{ are the dielectric, magnetoelastic, and magnetic permeability tensors, respectively; } E_k \text{ and } H_k \text{ are the electrical field and magnetic field vectors, respectively.}
\end{align*}
\]

The incremental form of Eq. (8a)-(8c) is expressed as
\[
\begin{align*}
\Delta \sigma_{ij} &= C_{ijkl} \Delta \epsilon_{kl} - e_{ijkl} \Delta E_k - q_{ijkl} \Delta H_k + \beta_{ij} \Delta \theta \\
- \Delta D_i &= -e_{ijkl} \Delta \epsilon_{kl} - k_{ij} \Delta E_k - a_{ik} \Delta H_k - p_{ij} \Delta \theta \\
- \Delta B_i &= -a_{ijkl} \Delta \epsilon_{kl} - a_{ik} \Delta E_k - \mu_{ik} \Delta H_k - m_{ij} \Delta \theta
\end{align*}
\]

2. Constitutive equations for metal

The incremental stress–strain relation of metals can be expressed as
\[
\Delta \sigma_{ij} = C_{ijkl} \Delta \epsilon_{kl} + \beta_{ij} \Delta \theta
\]

\[
\begin{align*}
- \Delta D_i &= -k_{ij} \Delta \epsilon_{kl} - a_{ik} \Delta H_k - p_{ij} \Delta \theta \\
- \Delta B_i &= -a_{ijkl} \Delta \epsilon_{kl} - \mu_{ik} \Delta H_k - m_{ij} \Delta \theta
\end{align*}
\]

\[C_{ijkl} \text{ are the components of the time independent fourth-order instantaneous tangent stiffness tensor which is the elastic stiffness tensor }\]
\[C_{ijkl} \text{ when the stress state of the material point is below yielding and the elastoplastic tangent stiffness tensor } C_{ijkl}^p \]
\[\text{when the stress state of the material point beyond yielding. According to the classical plasticity theory, the } C_{ijkl}^p \text{ is given by}
\]
\[C_{ijkl}^p = \left( C_{ijkl} \frac{\partial f}{\partial \sigma_{mn}} \left( \frac{\partial f}{\partial \sigma_{pq}} \right) C_{ijkl}^p \right) \left( \frac{\partial f}{\partial \sigma_{mn}} \left( \frac{\partial f}{\partial \sigma_{pq}} \right) C_{ijkl}^p \right)
\]

where \( f \) and \( p \) in Eq. (8a)-(8c) are yielding function and effective plastic strain, respectively.

3. Micromechanics formulations

Consider the smart composites with periodic microstructure as shown in Fig. 1. Two coordinate systems including \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \) are adopted to facilitate the micromechanics formulations. We use \( x_i \) as the global coordinates to describe the macroscopic structure and \( y_i \) parallel to \( x_i \) as the local coordinates to describe the UC (here and throughout the paper, Latin indices assume 1, 2, and 3 and repeated indices are summed over their range except where explicitly indicated). We choose the origin of the local coordinate system \( y_i \) to be the geometric center of UC.

3.1. General incremental formulations of the smart composite and its constituents

The incremental formulations of polymer, piezoelectric-piezomagnetic materials, and metal described by Eqs. (7a)-(7c), (9a)-(9c), and (10a)-(10c) can be extended to the following general incremental formulations:

\[
\Delta \sigma_{ij}(t) = M_{ijkl}(t) \Delta \epsilon_{kl}(t) - e_{ijkl}(t) \Delta E_k(t) - q_{ijkl}(t) \Delta H_k(t) + \beta_{ij}(t) \Delta \theta(t) + \omega(t)
\]

\[
- \Delta D_i(t) = -e_{ijkl}(t) \Delta \epsilon_{kl}(t) - k_{ij}(t) \Delta E_k(t) - a_{ik}(t) \Delta H_k(t) - p_{ij}(t) \Delta \theta(t) - m_{ij}(t)
\]

\[
- \Delta B_i(t) = -a_{ijkl}(t) \Delta \epsilon_{kl}(t) - a_{ik}(t) \Delta E_k(t) - \mu_{ik}(t) \Delta H_k(t) - m_{ij}(t) \Delta \theta(t) - \Psi(t)
\]
\[
\begin{align*}
\mathbf{K}_i(t) &= \begin{cases} 
K_{ik}(t) & \text{for polymer materials} \\
K_{ik} & \text{for piezoelectric materials} \\
K_{ik} & \text{for piezomagnetic materials} \\
K_{ik} & \text{for metal} 
\end{cases}, \\
\mathbf{\mu}_i(t) &= \begin{cases} 
\mu_{ik} & \text{for polymer materials} \\
\mu_{ik} & \text{for piezoelectric materials} \\
\mu_{ik} & \text{for piezomagnetic materials} \\
\mu_{ik} & \text{for metal} 
\end{cases}, \\
\mathbf{\rho}_i(t) &= \begin{cases} 
\gamma_{ij}(t) & \text{for polymer materials} \\
\gamma_{ij} & \text{for piezoelectric materials} \\
\gamma_{ij} & \text{for piezomagnetic materials} \\
\gamma_{ij} & \text{for metal} 
\end{cases}, \\
\mathbf{\omega}_i(t) &= \begin{cases} 
\omega_{ij}(t) & \text{for polymer materials} \\
\omega_{ij} & \text{for piezoelectric materials} \\
\omega_{ij} & \text{for piezomagnetic materials} \\
\omega_{ij} & \text{for metal} 
\end{cases}, \\
\mathbf{\sigma}_i(t) &= \begin{cases} 
\sigma_{ij}(t) & \text{for polymer materials} \\
\sigma_{ij} & \text{for piezoelectric materials} \\
\sigma_{ij} & \text{for piezomagnetic materials} \\
\sigma_{ij} & \text{for metal} 
\end{cases}, \\
\mathbf{\Psi}_i(t) &= \begin{cases} 
\Psi_{ij}(t) & \text{for polymer materials} \\
\Psi_{ij} & \text{for piezoelectric materials} \\
\Psi_{ij} & \text{for piezomagnetic materials} \\
\Psi_{ij} & \text{for metal} 
\end{cases}.
\end{align*}
\]

In Eq. (15), \( \mathbf{\Psi} \) is a 12 \times 11 matrix and expressed as
\[
\mathbf{\Psi} = \begin{bmatrix} \omega \\ \psi \end{bmatrix}
\]
where \( \omega \) is a 6 \times 1 submatrix for \( \omega_{ij}(t) \) coefficients; \( \psi \) is a 3 \times 1 submatrix for \( \sigma_{ij}(t) \) coefficients; and \( \psi \) is a 3 \times 1 submatrix for \( \Psi_{ij}(t) \) coefficients. The effective instantaneous thermo-electro-magneto-viscoelastic-like coefficients of smart composite materials can be defined in the following two ways:
\[
\Delta \mathbf{X} = R^T \Delta \mathbf{Y} + \eta^* \Delta \mathbf{N}(t) + \mathbf{\Psi}
\]
(19)
\[
\frac{1}{2} \oint \left[ \Delta \mathbf{Y}^T \mathbf{R} \Delta \mathbf{Y} + \Delta \mathbf{Y}^T \eta \Delta \mathbf{N}(t) + \Delta \mathbf{Y}^T \mathbf{\Sigma} \mathbf{\Psi} + \frac{1}{2} \mathbf{\Gamma} \mathbf{\Psi}^T \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} \right] d\Omega
= \frac{1}{2} \Delta \mathbf{Y}^T R^T \Delta \mathbf{Y} + \frac{1}{2} \Delta \mathbf{Y}^T \mathbf{\Sigma} \mathbf{\Psi} + \frac{1}{2} \mathbf{\Gamma} \mathbf{\Psi}^T \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} (20)
\]
where \( G \) is the energy change per unit temperature; \( c_v \) is the specific heat per unit volume at constant volume; \( T_0 \) is the reference temperature at which the constituent material is stress free; \( h_v \) is the energy change similar to \( c_v \). In Eqs. (19)–(20), “over-bar” indicates variables which are used in the macroscopic analysis of homogenized materials, and superscripts “*” denote the effective properties whose calculations are determined by the micromechanics model one employs. \( \Omega \) is the volume of unit cell.

3.2. \textit{VAMUCH model}

The total change of potential energy of the composites can be formulated as
\[
\Pi = \int \left[ \frac{1}{2} \Delta \mathbf{Y}^T \mathbf{R} \Delta \mathbf{Y} + \Delta \mathbf{Y}^T \eta \Delta \mathbf{N}(t) + \Delta \mathbf{Y}^T \mathbf{\Sigma} \mathbf{\Psi} + \frac{1}{2} \mathbf{\Gamma} \mathbf{\Psi}^T \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} \right] d\Omega
\]
(21)
with
\[
\Delta \mathbf{N}(t; \mathbf{x}; \mathbf{y}) = \frac{1}{2} \left[ \frac{\partial \Delta \mathbf{u}_i(t; \mathbf{x}; \mathbf{y})}{\partial y_j} + \frac{\partial \Delta \mathbf{u}_i(t; \mathbf{x}; \mathbf{y})}{\partial y_j} \right] \equiv \Delta \mathbf{u}_{i;j}(t)
\]
(22)
\[
\Delta \mathbf{E}_i(t; \mathbf{x}; \mathbf{y}) = - \frac{\partial \Delta \mathbf{\sigma}^o(t; \mathbf{x}; \mathbf{y})}{\partial y_i}
\]
(23)
\[
\Delta \mathbf{H}_i(t; \mathbf{x}; \mathbf{y}) = - \frac{\partial \Delta \mathbf{\phi}^o(t; \mathbf{x}; \mathbf{y})}{\partial y_i}
\]
(24)
where \( \Delta \mathbf{u}_i, \Delta \mathbf{\sigma}^o, \), and \( \Delta \mathbf{\phi}^o \) are the increments of displacement vector, electrical potential, and magnetic potential, respectively. Since \( \Delta \mathbf{u}_i, \Delta \mathbf{\sigma}^o, \), and \( \Delta \mathbf{\phi}^o \) are continuous functions at the interfaces between adjacent unit cells, we have
\[
\Delta \mathbf{u}_i(t; x_1, x_2, x_3; d_1/2, y_2, y_3) = \Delta \mathbf{u}_i(t; x_1, x_2, x_3; -d_1/2, y_2, y_3)
\]
\[
\Delta \mathbf{u}_i(t; x_1, x_2, x_3; y_1, d_2/2, y_3) = \Delta \mathbf{u}_i(t; x_1, x_2, x_3; y_1, -d_2/2, y_3)
\]
\[
\Delta \mathbf{u}_i(t; x_1, x_2, x_3; y_1, y_2, d_3/2) = \Delta \mathbf{u}_i(t; x_1, x_2, x_3; y_1, y_2, -d_3/2)
\]
(25)
\[
\Delta \mathbf{\phi}^o(t; x_1, x_2, x_3; d_1/2, y_2, y_3) = \Delta \mathbf{\phi}^o(t; x_1, x_2, x_3; -d_1/2, y_2, y_3)
\]
\[
\Delta \mathbf{\phi}^o(t; x_1, x_2, x_3; y_1, d_2/2, y_3) = \Delta \mathbf{\phi}^o(t; x_1, x_2, x_3; y_1, -d_2/2, y_3)
\]
\[
\Delta \mathbf{\phi}^o(t; x_1, x_2, x_3; y_1, y_2, d_3/2) = \Delta \mathbf{\phi}^o(t; x_1, x_2, x_3; y_1, y_2, -d_3/2)
\]
(26)
\[
\Delta \mathbf{\phi}^m(t; x_1, x_2, x_3; d_1/2, y_2, y_3) = \Delta \mathbf{\phi}^m(t; x_1, x_2, x_3; -d_1/2, y_2, y_3)
\]
(27)
\[ \Delta \varphi^m(t; x_1, x_2, x_3; y_1, d_2/2, y_3) = \Delta \varphi^m(t; x_1, x_2 + d_2, n_3; y_1, -d_2/2, y_3) \]
\[ \Delta \varphi^m(t; x_1, x_2, x_3; y_1, y_2, d_3/2) = \Delta \varphi^m(t; x_1, x_2, x_3 + d_3; y_1, y_2, -d_3/2) \]

\[ (\mathbf{f}) = -\frac{1}{\Omega} \int_{\Omega} \mathbf{f} \, d\Omega \tag{28} \]

with \( \Omega \) denoting the unit cell domain.

In Eq. (28) and throughout this paper, the angle bracket denotes average over the domain of the UC, that is

In order to avoid the difficulties associated with the unknowns introduced by the Lagrange multipliers, we reformulate the stationary problem of the functional in Eq. (39) as the minimum value of the following functional:

\[ \Pi_{\Omega} = \frac{1}{\Omega} \int_{\Omega} \left[ 2 \Delta \varphi^m(t; \mathbf{x}; \mathbf{y}) + \Delta \varphi^m(t; \mathbf{x}; \mathbf{y}) + \frac{1}{2} \frac{\Delta \varphi^m(t; \mathbf{x}; \mathbf{y})}{T_0} + \frac{1}{2} \frac{\Delta \varphi^m(t; \mathbf{x}; \mathbf{y})}{T_0} \right] d\Omega \tag{45} \]

under the following constraints:

\[ X_{i,j} = X_{i,j} \]
\[ \varepsilon_{i,j} = \varepsilon_{i,j} \]
\[ \varepsilon_{m,j} = \varepsilon_{m,j} \]
\[ \varepsilon_{m,j} = \varepsilon_{m,j} \]

for \( i,j = 1, 2, 3 \)

Introduce the following matrix form:

\[ \begin{pmatrix} \varepsilon_{i,j} \\ \varepsilon_{m,j} \end{pmatrix} \equiv \Gamma_{h} X \tag{46} \]

where \( \Gamma_{h} \) is an operator matrix and \( X \) is a column matrix containing the components of the fluctuation functions. If we discretize \( X \) using the finite elements as

\[ X(x_i; y_j) = \sum_{x_i} X(x_i) \]

where \( S \) representing the shape functions (in assemble sense excluding the constrained node and slave nodes) and \( X \) column matrix of the nodal value of the fluctuation functions for all active nodes. Substituting Eqs. (46) and (47) into Eq. (45), we obtain a discretized version of the functional as

\[ \Pi_{\Omega} = \frac{1}{\Omega} \sum_{x_i} \left( X^T \mathbf{E} X + 2X^T D_{hh} X + 2X^T D_{hh} \Delta \varphi^m(t) \right) \]

\[ + 2 \Delta \varphi^m(t) D_{hh} \Delta \varphi^m(t) + 2X^T D_{hh} \Delta \varphi^m(t) + 2X^T D_{hh} \Delta \varphi^m(t) + 2X^T D_{hh} \Delta \varphi^m(t) \]

\[ + D_{hh} \frac{\Delta \varphi^m(t)}{T_0} + D_{cc} \]
where
\[ E = \int_{\Omega} (\gamma_h \Sigma) \Sigma d \Omega \quad D_{\text{ho}} = \int_{\Omega} (\gamma_h \Sigma) \Sigma d \Omega \]
\[ D_{\nu} = \int_{\Omega} \nu_{\theta} d \Omega \quad D_{\text{ho}} = \int_{\Omega} (\gamma_h \Sigma) \Sigma d \Omega \]
\[ D_{\eta} = \int_{\Omega} \eta_{\theta} d \Omega \quad D_{\text{ho}} = \int_{\Omega} (\gamma_h \Sigma) \Sigma d \Omega \]
\[ D_{\sigma} = \int_{\Omega} \sigma_{\theta} d \Omega \quad D_{\text{ho}} = \int_{\Omega} (\gamma_h \Sigma) \Sigma d \Omega \]
\[ D_{\text{ho}} = \int_{\Omega} \text{ho} d \Omega \quad D_{\text{ho}} = \int_{\Omega} (\gamma_h \Sigma) \Sigma d \Omega \]
(49)

Minimizing \( \Pi_{\Omega} \) in Eq. (48), we obtain the following linear system:
\[ \mathbf{E} \mathbf{X} = - D_{\text{ho}} \Delta \mathbf{Y} - D_{\nu} \Delta \mathbf{\theta} - D_{\text{ho}} \]
(50)

The fluctuation function \( \mathbf{X} \) is linearly proportional to \( \Delta \mathbf{Y} \) and \( \Delta \mathbf{\theta} \), which means the solution can be written as
\[ \mathbf{X} = x_0 \Delta \mathbf{Y} + x_\mathbf{\theta} \Delta \mathbf{\theta} + \mathbf{X}_c \]
(51)

Substituting Eq. (51) into Eq. (48), we can calculate the free energy density of the UC as
\[ \Pi_{\Omega} = \frac{1}{2} \Delta \mathbf{Y}^T \mathbf{R} \Delta \mathbf{Y} + \Delta \mathbf{Y}^T \mathbf{\eta} \Delta \mathbf{\theta} + \Delta \mathbf{Y}^T \mathbf{K}^* + \frac{1}{2} \mathbf{C}^* \Delta \mathbf{\theta}^2 + \frac{1}{2} \mathbf{R}^* \Delta \mathbf{\theta} \]
(52)

with
\[ \mathbf{R}^* = \frac{1}{\Omega^2} \left[ \delta_{\theta} \mathbf{D}_{\text{ho}} + D_{\nu} \right] \]
\[ \mathbf{\eta}^* = \frac{1}{\Omega^2} \left[ D_{\nu} \mathbf{\chi} + \chi_0 \mathbf{D}_{\text{ho}} + D_{\nu} \right] \]
\[ \mathbf{K}^* = \frac{1}{\Omega^2} \left[ D_{\text{ho}} \mathbf{\chi} + \chi_0 \mathbf{D}_{\text{ho}} + D_{\nu} \right] \]
\[ \mathbf{C}^* = \frac{1}{\Omega^2} \left[ \chi_0 \mathbf{D}_{\text{ho}} + \mathbf{D}_{\nu} \right] \]
\[ \mathbf{R}^* = \frac{1}{\Omega^2} \left[ \chi_0 \mathbf{D}_{\text{ho}} + D_{\nu} \right] \]
(53)

where \( \mathbf{R}^* \) is a 12 \times 12 effective material matrix containing instantaneous multiphysics material properties; \( \mathbf{\eta}^* \) is a 12 \times 1 effective matrix containing the effective instantaneous second order thermal stress tensor \( \rho_{\theta}^* \), the effective pyroelectric vector \( p_{\theta}^* \) and the effective pyromagnetic vector \( m_{\theta}^* \); \( \mathbf{K}^* \) is a 12 \times 1 effective matrix containing the effective instantaneous second order moment \( \mathbf{a}_{\theta}^* \), the effective vector \( \mathbf{m}_{\theta}^* \) and the effective vector \( \mathbf{\psi}_{\theta}^* \).

After having uniquely determined the fluctuation functions, we can recover the local displacement, electric potential, and magnetic potential as
\[ \left\{ \begin{array}{c}
\Delta \mathbf{u}_1 \\
\Delta \mathbf{u}_2 \\
\Delta \mathbf{u}_3 \\
\Delta \mathbf{\phi}_{\theta}^m \\
\Delta \mathbf{\phi}_{\theta}^S
\end{array} \right\} = \left\{ \begin{array}{c}
\Delta \mathbf{v}_1 \\
\Delta \mathbf{v}_2 \\
\Delta \mathbf{v}_3 \\
\Delta \mathbf{\psi}_{\theta}^m \\
\Delta \mathbf{\psi}_{\theta}^S
\end{array} \right\} + \left\{ \begin{array}{c}
\mathbf{y}_1 \\
\mathbf{y}_2 \\
\mathbf{y}_3
\end{array} \right\} + \mathbf{S} \mathbf{X}
\]
(54)

where \( \mathbf{S} \) is different from \( S \) and \( \mathbf{X} \) is different from \( X \) due to the recovery of slave nodes and the constrained node. The increments of the local strain field, local electrical field, and local magnetic field can be recovered as
\[ \Delta \mathbf{Y} = \Delta \mathbf{Y} + \gamma_\Sigma \mathbf{S} \mathbf{X} \]
(55)

Finally, the increments of the local stress field, electrical displacement field, and magnetic flux density can be recovered straightforwardly using the 3D constitutive relations for the constituent material as
\[ \Delta \mathbf{X} = \mathbf{R} \Delta \mathbf{Y} + \eta \Delta \mathbf{\theta} + \mathbf{J} \]
(56)

The simulations of the time-dependent and non-linear thermo-electro-magneto-viscoelastic-plastic response of smart composites are performed using an incremental procedure based on Eq. (19). Once the \( \mathbf{R}^* \), \( \mathbf{\eta}^* \), and \( \mathbf{J}^* \) have been determined at the current mechanical loading, electrical charge, or magnetic charge, one can determine the current values of variables from previous values and increments according to
\[ \mathbf{X}_{\text{current}} = \mathbf{X}_{\text{previous}} + \Delta \mathbf{X} \]
(57a)
\[ \mathbf{V}_{\text{current}} = \mathbf{V}_{\text{previous}} + \Delta \mathbf{V} \]
(57b)

The simulations can be readily performed without applying various boundary conditions as those are carried out using finite element unit cell procedures.

4. Numerical examples

4.1. Material properties of constituents

Aluminum: The material properties of aluminum are presented in Table 1. The mechanical behavior of the aluminum is described using rate-independent elasto-plastic model expressed in Eqs. (10) and (11). Isotropic linear hardening is assumed for the aluminum.

Piezoelectric and Piezomagnetic material: Table 2 presents the material properties of piezoelectric material (BaTiO3) and piezomagnetic material (CoFe2O4).

<table>
<thead>
<tr>
<th>Material properties of aluminum core.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus ( E ) (MPa)</td>
</tr>
<tr>
<td>70,000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BaTiO3 (piezoelectric)</th>
<th>CoFe2O4 (piezomagnetic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{11} ) (GPa)</td>
<td>162</td>
</tr>
<tr>
<td>( C_{12} ) (GPa)</td>
<td>78</td>
</tr>
<tr>
<td>( C_{13} ) (GPa)</td>
<td>77</td>
</tr>
<tr>
<td>( C_{22} ) (GPa)</td>
<td>166</td>
</tr>
<tr>
<td>( C_{33} ) (GPa)</td>
<td>43</td>
</tr>
<tr>
<td>( k_{11} ) (C/V m)</td>
<td>12.6 \times 10^{-9}</td>
</tr>
<tr>
<td>( k_{13} ) (C/V m)</td>
<td>11.2 \times 10^{-9}</td>
</tr>
<tr>
<td>( \mu_{11} ) (N m^2/C^2)</td>
<td>0.1 \times 10^{-4}</td>
</tr>
<tr>
<td>( \mu_{13} ) (N m^2/C^2)</td>
<td>0.05 \times 10^{-4}</td>
</tr>
<tr>
<td>( e_{11} ) (C/m^2)</td>
<td>18.6</td>
</tr>
<tr>
<td>( e_{12} ) (C/m^2)</td>
<td>-4.4</td>
</tr>
<tr>
<td>( e_{33} ) (C/m^2)</td>
<td>11.6</td>
</tr>
<tr>
<td>( \varrho_{11} ) (N/A m)</td>
<td>0</td>
</tr>
<tr>
<td>( \varrho_{12} ) (N/A m)</td>
<td>0</td>
</tr>
<tr>
<td>( \varrho_{33} ) (N/A m)</td>
<td>0</td>
</tr>
<tr>
<td>( a_{11} ) (1/C)</td>
<td>6.4 \times 10^{-6}</td>
</tr>
<tr>
<td>( a_{12} ) (1/C)</td>
<td>15.7 \times 10^{-6}</td>
</tr>
<tr>
<td>( a_{33} ) (1/C)</td>
<td>15.7 \times 10^{-6}</td>
</tr>
</tbody>
</table>
can be expressed using Prony series as
\[ E(t) = E_0 \left(1 - \sum_{k=1}^{n} g_k \left(1 - e^{-t/\tau_k}\right)\right) \] (58)
where \( E_0 \) is the instantaneous Young’s modulus; \( g_k \) is dimensionless modulus and \( \tau_k \) is the time relaxation material parameter. For simplicity, we considered a special case, namely, \( n = 1, g_1 = 0.5, \) and \( \tau_1 = 30, \) such that Eq. (58) is reduced to
\[ E(t) = 0.5E_0 \left(1 - e^{-t/\rho}\right) = A + Be^{-t/\rho} \] (59)
where \( E_0 = 8000 \text{ MPa} \) and \( \rho = 30, \) then \( A = B = 4000 \text{ MPa}. \) The dielectric coefficient and magnetic permeability of the polymer are assumed to be constant and assigned as \( k = 0.1 \times 10^{-9} \text{ C/(V.m)} \) and \( \mu = 0.01 \times 10^{-4} \text{ Ns}^2/\text{C}^2, \) respectively. The thermal expansion of the polymer material are kept constant as \( \alpha = 54 \times 10^{-6} \text{ °C}^{-1}. \)

The time-scale shift factor \( a_t \) in Eq. (3) is determined by empirical relationship of Williams–Landel–Ferry (WLF) [19],
\[ \log a_t(T) = \frac{C_1(T - T_0)}{C_2 + (T - T_0)} \] (60)
where \( C_1 \) and \( C_2 \) are material constants determined through least squares fitting. In this example, the values of \( C_1 \) and \( C_2 \) are set as \( C_1 = 4.92 \) and \( C_2 = 215.0. \)

\( T_0 \) in Eq. (60) is the reference temperature and the temperature \( T \) at time \( t \) is given by
\[ T = T_0 + \theta = T_0 + \alpha_0 t \] (61)
where \( \alpha_0 \) is the temperature change rate.

Therefore, the reduced time \( \xi(t + \Delta t), \xi(t), \) and \( \xi(t) \) are given by
\[ \xi(t + \Delta t) = \int_0^{t+\Delta t} 10^{C_1\xi(t)}/C_2 dt' \]
\[ \xi(t) = \int_0^t 10^{C_1\xi(t)}/C_2 dt' \]
\[ \xi(t) = \int_0^t 10^{C_1\xi(t)}/C_2 dt' \] (62)

The stress relaxation stiffness matrix \( [L_{ij}(t)] \) in Eq. (7a) is obtained as
\[ [L_{ij}(t)] = f [W] \] (63)
where the coefficient \( f \) is computed using Simpson’s rule of numerical integration as
\[ f = \frac{t}{6}(fL + 4fm + fmu) \] (64)
where
\[ fL = A + Be^{-t/\rho} \]
\[ fmu = A + Be^{-\xi(t)/2} \]
\[ fm = A + Be^{-\xi(t)/\rho} \]

The matrix \([W]\) in Eq. (63) is given by
\[ [W] = \frac{1}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 1-\nu & 1-\nu & 0 & 0 & 0 \\ \nu & 1-\nu & 0 & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (65)
with \( \nu \) being the Poisson’s ratio of the polymer, which is assumed to be constant \( \nu = 0.4. \)

The matrix \([\gamma(t)] \) in Eq. (7a) is obtained as
\[ [\gamma(t)] = [L_{ij}(t)] [\alpha] \] (66)
where \( [\alpha] \) is a column matrix containing thermal expansion coefficients of the polymer materials.

The coefficient matrix of \( \alpha(t) \) in Eq. (7a) is calculated as
\[ [\alpha(t)] = \frac{B}{\alpha} \sqrt{\int_{t-\Delta t}^{t} \left[ \int_{t}^{t+\Delta t} \left( e^{-t} - e^{-\xi(t)\tau} \right) [W][\xi] d\xi \right] d\Delta t} \] (67)
where \( [\Delta \xi(i)] \) is \( 6 \times 1 \) column matrix containing strain increments during the \( i \)th time step \( \Delta t; \Delta \theta(i) \) are the increment of temperature change during the \( i \)th time step \( \Delta t; \) and \( n = t/\Delta t. \)

4.2. Model verification

Let us firstly calculate the stress–strain hysteresis loop of the polymer matrix using VAMUCH and ABAQUS. The polymer was simultaneously applied with cyclic stress loading and various temperature changes as shown in Fig. 2. The temperature changes increase linearly and uniformly up to 20 °C, 40 °C, 60 °C, and 80 °C, respectively. The corresponding responses of the polymer are illustrated in Fig. 3 in which the stress–strain hysteresis loop of the polymer without thermal effects is also plotted for the sake of comparison. Since the VAMUCH predictions are identical to ABAQUS results, only VAMUCH results are illustrated in Fig. 3 from which one can see that the finally remained strains increase as the temperature changes increase after reverse unloading.

![Fig. 2](image-url)
Let us consider an aluminum fiber reinforced polymer matrix composites with the volume fraction of aluminum fiber being \( v_{o f} = 0.4 \). The aluminum fiber is of circular shape and in square array. The composite is simultaneously applied with the cyclic stress loading (along fiber direction and transverse direction, respectively) and various temperature changes as shown in Fig. 2. The stress–strain hysteresis curves of aluminum fiber polymer matrix composite due to axial and transverse cyclic loadings coupled with temperature change are plotted in Fig. 4 from which one can observe that there are excellent agreements between the predictions of VAMUCH and ABAQUS.

Next consider a piezoelectric (BaTiO3) fiber reinforced aluminum matrix composite with the volume fraction of BaTiO3 fiber being \( v_{o f} = 0.4 \). The composite was simultaneously applied with temperature changes shown in Fig. 2(b) and electrical field \( E_1 \) along the fiber direction. Fig. 5 shows the variation of the effective induced axial strain \( \varepsilon_{11} \) of traction-free piezoelectric (BaTiO3) fiber reinforced aluminum matrix composite with the applied electrical field \( E_1 \) at various temperature changes. It can be seen that VAMUCH and ABAQUS provide identical predictions. VAMUCH can also accurately recover the local fields based on the global responses and the recovery relations obtained from the micromechanics analysis. Fig. 6 illustrates the contour plot of local stress component \( \sigma_{22} \) (predicted by VAMUCH) and the comparison of the distributions of \( \sigma_{22} \) along the \( y_2 \) axis predicted by VAMUCH and ABAQUS. The results were calculated when the electrical field \( E_1 \) and temperature change increased up to 1.96 MV/m and 60 °C, respectively. One can see from Fig. 6 that the recovery capability of VAMUCH is validated by ABAQUS results.

4.3. Effective responses of aluminum core piezoelectric and piezomagnetic fiber (APPF) composites

In this section, the effective responses of aluminum core piezoelectric and piezomagnetic fiber (APPF) reinforced polymer matrix composite whose microstructure is shown in Fig. 7 were investigated using VAMUCH. The volume fractions of polymer matrix, aluminum core, and piezoelectric material are 0.55, 0.3, and 0.05, respectively. We firstly considered the effects of strain rates on the uniaxial tension behavior. Fig. 8 presents the effective uniaxial tension stress–strain curves of APPF reinforced polymer matrix composite at various strain rates while all electrical fields and magnetic fields...
are kept zero. Obviously, the tension behaviors of the composite are rate dependent due to the time dependent behavior of the polymer matrix while the influences of strain rates are more pronounced on the transverse tension than on axial tension.

Let us consider the thermal induced axial stress $\sigma_{11}$ of APPF polymer matrix composites when all mechanical strains, electric fields, and magnetic fields are kept as zero. This is shown in Fig. 9, which demonstrates that the magnitude of induced axial stress increases with the increase of the temperature change rate. The variation of slope of the curve of induced stress $\sigma_{11}$ vs temperature change is due to the modulus relaxation of the polymer matrix.

When all mechanical strains are equal to zero, the induced axial stresses $\sigma_{11}$ of APPF polymer matrix composites are generated by axial electric field $E_1$ and axial magnetic field $H_1$ are presented in Figs. 10 and 11, respectively. It can be seen that the rate of $E_1$ and $H_1$ do not have significant influences on the induced longitudinal overall stress $\sigma_{11}$.

The deformation of smart composites induced by electric input can be utilized for sensing applications. Figs. 12 and 13 illustrate such responses of traction free APPF reinforce polymer matrix.

Fig. 6. The recovered local stress of piezoelectric (BaTiO$_3$) fiber reinforced aluminum matrix composite: (a) Contour plot of stress component $\sigma_{22}$ (Pa) and (b) the distribution of $\sigma_{22}$ along the $y_2$ axis.

Fig. 7. Microstructure of smart composites consisting of aluminum core, piezomagnetic material, piezoelectric material, and polymer matrix.

Fig. 8. Effective uniaxial tension stress–strain curves of APPF reinforced polymer matrix composite at various strain rates while all electrical fields and magnetic fields are kept zero: (a) axial tension $\sigma_{11}$ vs $\varepsilon_{11}$ and (b) transverse tension $\sigma_{22}$ vs $\varepsilon_{22}$.
composites. These two figures also clearly illustrate that the yielding of aluminum core causes the change of the slope of the lines of $\varepsilon_{11}$ vs $E_1$ and $\varepsilon_{11}$ vs $H_1$ when the axial electrical field $E_1$ and axial magnetic field $H_1$ increased up to certain values.

5. Conclusions

We have developed a general-purpose micromechanical model that is capable of predicting the fully coupled thermo-electro-magneto-viscoelastic-plastic response of smart composites. In view of the time dependent characteristics and non-linearity of the composite, instantaneous tangential electro-magneto-mechanical matrices associated with an incremental procedure were established. The proposed model can efficiently capture the rate-dependent and non-linear behavior of multiphase smart composite consisting of linear viscoelastic materials, piezoelectric materials, piezomagnetic materials, and metallic phases.

References